Discussion 6A Recap

Tyler Zhu

March 15, 2020

Congrats! You've officially made it past the discrete mathematics portion of the course. Now we're moving onto counting and probability (really the same subject, except you divide by the total in the latter). If you were lost before, it's time to catch up now, as getting lost in these first few weeks would be *really* bad...

1 Counting

If I have k categories with n_i items in each, then the number of ways I can pick one item from each category is $n_1 \times n_2 \times \cdots \times n_k$. This is the **First Rule of Counting**.

One important application is to counting the number of permutations.

Question (Permutations). How many ways can I rearrange n distinct books on a bookshelf?

Our categories are the number of books which can go first, then how many can go second, third, and so on. There are n choices for the first book, and once we pick it, we only have $n-1$ choices for the second book. This continues until we only have one choice for the last book. Hence, in total, the total number of ways is $n \times (n-1) \times \cdots \times 2 \times 1$.

This quantity comes up so often that we have a special name for it: the **factorial**.

Definition 1 (Factorial). The number of permutations of n objects is defined to be $n!$ (n *factorial*), and $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

Another common question that comes up is the number of subsets of a given size.

Question (Subsets). How many ways can I choose k items from n without regards to order?

This is very similar to counting permutations. We have n choices for our first item, then $n-1$ choices for our second item, up until $n-k+1$ choices for our kth item. But this includes an implicit ordering of our choices; choosing item A and then item B becomes different from choosing item B and then item A, which is not what we want.

To deal with this overcount, we divide by the total number of times a subset is counted if order matters. For any given subset, all of the $k!$ different permutations of the items are counted as different, so we're over counting by a factor of $k!$. Our final answer is just

$$\frac{n \times (n-1) \times \cdots \times (n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

This quantity is just as important as the factorial, so we also have a special name for it: the **binomial coefficient**.

Definition 2 (Binomial Coefficient). The number of ways to pick k distinct items from n total items (where order doesn't matter) is defined to be $\binom{n}{k}$ (read n *choose* k), and $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

The idea that we can count the number of assignments where order doesn't matter by pretending order matters first, then dividing out by the overcount is known as the **Second Rule of Counting**.

As a summary, here are the four main ways to sample k items from a group of n under different conditions. See if you can figure out why each entry is true (one of them is more advanced than the others; read ahead if you want to understand).

	sampling w/ replacement	sampling w/o replacement
order matters	n^k	$\frac{n!}{k!}$
order doesn't matter	$\binom{n+k-1}{k-1}$	$\binom{n}{k}$

Figure 1: These answers should come naturally to you. Don't memorize them!

2 Anagrams

Traditionally, anagram problems do trip people up, so I wanted to present an explanation that I learned from another TA last semester that I thought was quite nice.

Example 2.1. How many different anagrams of GHOST are there if G is on the left of H (not necessarily neighbors)?

Proof. Since the G's and H's have weird restrictions, replace them with ?'s for now. Suppose we want to know how many different anagrams of ??OST there are. There are 5! ways to rearrange this assuming everything is distinct, but the ?'s are indistinguishable, so we're overcounting by 2!. In total, that's $\frac{5!}{2!}$ ways.

Now say we have some ordering T?S?O. How do we know where the G and H go? We know that the first ? must be G, and the second one must be H. So for every ordering of $??OST$, we get exactly one anagram of GHOST where G appears before H. Hence, our total is $\frac{5!}{2!}$. □

3 Tips

- Don't memorize formulas in counting. You should be able to derive any of them on the spot and tweak them for different scenarios.
- It's also not enough to just know the right answers. It's more important to know why the wrong answers are wrong. So ask!
- A good way to double check your answers is see if you can arrive at the same answer counting two different ways.
- Keep track of when order matters and when it doesn't (sock problem is a good example of this).
- Find a way to fix orderings when counting. For example, if we're trying to find the number of ways to pair up 20 different socks, create pairs by fixing the smallest numbered sock not yet chosen as the first sock. There is an unambiguous ordering both within each pairs and between all of the pairs; the smaller number in each pair comes first, and all of the pairs are ordered by increasing smallest sock. This let's us compute the result as $19 \times 17 \times \cdots \times 1$. The same idea applies for finding the number of permutations of GHOST where G comes before H.