Discussion 2B Recap

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1 Terminology

A **graph** G consists of **vertices** or **nodes**, the points, and **edges**, the lines. We frequently write $V(G)$ and $E(G)$ for the vertex and edge sets of G respectively, and call $|V(G)|$ and $|E(G)|$ the **order** and **size** of the graph respectively.

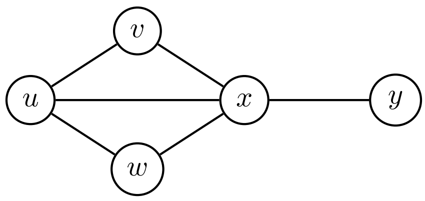

Figure 1: A graph of order 5 and size 6.

In Figure 1, $V(G) = \{u, v, w, x, y\}$ and $E(G) = \{uv, ux, uw, vx, wx, xy\}$. If $e = uv$ is an edge of G, then u and v are said to be **joined** by the edge e. In this case, u and v are referred to as **neighbors** of each other.

For a connected graph G, any open trail that contains every edge of G is an **Eulerian trail**. If G contains a closed Eulerian trail, it is **Eulerian**

Theorem 1. A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.

With this, we can easily characterize graphs possessing an Eulerian trail.

Corollary 2. A connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at another.

There is an analog to Eulerian trails. A path in a graph G that contains every vertex of G is called a **Hamiltonian path**.[1] Unfortunately, these are much less well-behaved.

2 Planarity Bounds

A graph G is called a **planar graph** if G can be drawn in the plane so that no two of its edges cross each other. If G is planar, then it divides the plane into pieces called **regions**. Recall that for any planar, connected graph G, if G has V vertices, E edges, and F faces, Euler's formula tells us that

$$V - E + F = 2.$$

You can imagine how hard it is to prove a graph to be non-planar: you can't possibly check every way of drawing the graph. Euler's Identity leads to some simple conditions on planarity.

[1] We studied Eulerian *trails* because such trails may need to have repeated vertices, unlike paths which necessarily have unique vertices.

We can derive many useful bounds using this formula that involve only two of the three quantities, which is helpful especially for showing that certain graphs are nonplanar. Usually we omit faces since those are tricky to quantify, so let's try to compare the number of faces to the number of edges.

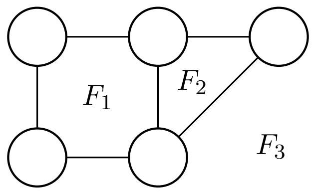

Figure 2: An example planar graph G_1 with its three faces marked.

Let's look at the graph G_1 shown in Figure 2. Notice that every edge is a part of exactly two faces. So we can count the number of edges also by looking at how many edges border each face. Let $|F_i|$ denote the number of edges that border the face F_i. Then,

$$\begin{aligned} 2E &= |F_1| + |F_2| + |F_3| \\ &= 4 + 3 + 5 \\ &\geq 3 + 3 + 3 \\ &= 3F \end{aligned}$$

where we used the very deep fact that every face is bordered by at least 3 edges (how else do you have a face?). Hence, we've arrived at the inequality $2E \geq 3F$. Now if we solve our formula for F and substitute, we get the all important bound

$$2 - V + E = F \leq \frac{2}{3}E \implies \boxed{E \leq 3V - 6}.$$

This is useful now as a method for showing a graph is nonplanar, as otherwise we'd have to draw every possible configuration of a graph and show all of them have crossings, which is neither feasible nor convincing. We can use this to show that K_5, the complete graph on 5 vertices, is nonplanar. Since it has 5 vertices and 10 edges, $10 \not\leq 3 \cdot 5 - 6 = 9$.

One extension of the above bound is to find a bound when G is triangle-free, i.e. no face is bounded by 3 edges. I'll leave that as an exercise for you. Recall that the graph with six vertices, where three vertices are connected to all three other vertices, is the complete bipartite graph $K_{3,3}$. We can use this new bound to show $K_{3,3}$ is nonplanar.

Exercise 2.1. Show that if G is a connected, planar, triangle-free graph, then $E \leq 2V - 4$. Use this to show that $K_{3,3}$ is nonplanar.

As a hint, we proved that all bipartite graphs have no odd tours. So what do we know about the presence of triangles in such a graph?

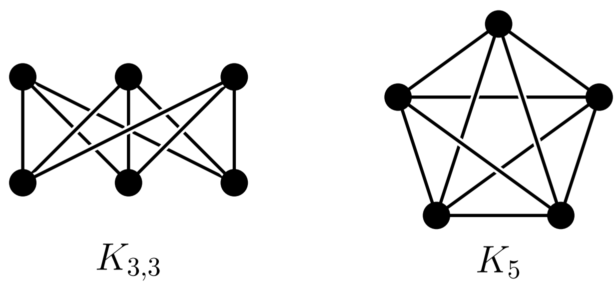

Figure 3: The Kuratowski graphs

We can already see how much trouble it can be to determine planarity for arbitrary graphs, let alone simple ones like $K_{3,3}$ and K_5. Surprisingly, the main enemies to planarity are precisely these two graphs, which leads to a simple check to see if a graph is planar, discovered in 1930. Before I can state Kuratowski's remarkable theorem, I need to state two notions.

A **subdivision** of a graph is constructed by replacing one edge by two edges with a vertex in between, and a **subgraph** is constructed by removing vertices or edges.

Theorem 3 (Kuratowski)**.** A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

In other words, G is planar if it is not possible to subdivide the edges of K_5 or $K_{3,3}$, and then possibly add edges or vertices, to get G.

2.1 To Take Home

Exercise 2.2 (Dis. Problem 2b)**.** Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G , there are at least two edges among them. Prove that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Hint: Proof by contradiction when $v = 7$. What do we know about groups of five vertices in a planar graph?

3 Induction on Edges and Vertices

Problem 3 from the discussion sometimes gives people trouble, so I'll try to explain it some more in depth. Here is the problem, paraphrased.

Problem 1. An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors.

(a) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors.

(b) Show that any tree with maximum degree $d \geq 1$ can be edge colored with d colors.

The solution is to use induction on the number of edges in the first question, and induction on the number of vertices in the second. I'm not going to re-explain the solution (you can find it online) but I'll try to explain why it's fine to induct this way.

Imagine if I had the following problem:

Question. Prove that any graph with maximum degree 4 can be edge colored with 7 colors.

Now induction, especially on the number of edges, seems like a viable approach. We're proving some statement about all graphs, so let's assume its true for a graph with m edges and prove it for one with $m + 1$ edges (both with max degree 4). What about this problem:

Question. Prove that any graph with maximum degree 10 can be edge colored with 19 colors.

See what I'm getting at? Just because we have a variable d in our original statement doesn't mean we need to induct over d. We can treat d as a constant and show the statement is true for all graphs given a specific d, which then shows the statement is true for all d.

Another analogy: we don't need to induct over our variable d much like how the number of vertices and edges are also variables, but we don't need to induct over both.