Discussion 10A Recap

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1 Variance

Recall the definition of expectation:

Definition 1 (Expectation). The expectation of a random variable X is

$$\mathbb{E}[X] = \sum_{x \in \mathcal{A}} x\mathbb{P}(X = x).$$

Today we will define another statistic of a random variable which will be very useful in our future analyses.

Definition 2 (Variance). The variance of a random variable X is

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \mathcal{A}} (x - \mathbb{E}[X])^2 \mathbb{P}(X = x).$$

In other words, the variance measures the average *squared deviations* away from the mean that our distribution takes on. We have to square the difference so that they don't cancel each other out.

Remember that expectation of a function of any random variable is an average of the values of the r.v. with the function applied, weighted by the probabilities, as one would expect. Formally,

$$\mathbb{E}[f(X)] = \sum_{x \in \mathcal{A}} f(x)\mathbb{P}(X = x).$$

This is sometimes referred to as the *law of the unconscious statistician*.

Variance has a number of important properties. The following is the most common form of variance used.

Theorem 3. For a random variable X, $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

Proof. We simply expand. Notice that

$$\begin{aligned}
\text{Var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= \mathbb{E}[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2] \\
&= \mathbb{E}[X^2] - \mathbb{E}[2X\mathbb{E}[X]] + \mathbb{E}[\mathbb{E}[X]^2] && \text{(by linearity of expectation)} \\
&= \mathbb{E}[X^2] - 2\mathbb{E}[X]^2 + \mathbb{E}[X]^2 && \text{(by linearity since } \mathbb{E}[X] \text{ is a constant)} \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2
\end{aligned}$$

□

Finally, we also have some basic linearity principles for variance (with and without independence).

Fact 4. $\text{Var}(cX) = c^2\text{Var}(X)$.

Fact 5. For independent r.v.'s X, Y, $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ and $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

2 Geometric Distributions

For a geometric distribution $X \sim \text{Geo}(p)$, we also have that $\mathbb{E}[X] = \frac{1}{p}$ and $\text{Var}(X) = \frac{1-p}{p^2}$.