Discussion 2B Recap

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1 Terminology

A **graph** G consists of **vertices** or **nodes**, the points, and **edges**, the lines. We frequently write $V(G)$ and $E(G)$ for the vertex and edge sets of G respectively, and call $|V(G)|$ and $|E(G)|$ the **order** and **size** of the graph respectively.

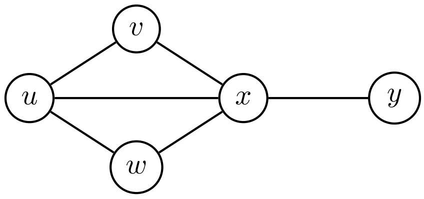

Figure 1: A graph of order 5 and size 6.

In Figure 1, $V(G) = \{u, v, w, x, y\}$ and $E(G) = \{uv, ux, uw, vx, wx, xy\}$. If $e = uv$ is an edge of G, then u and v are said to be **joined** by the edge e. In this case, u and v are referred to as **neighbors** of each other.

For a connected graph G, any open trail that contains every edge of G is an **Eulerian trail**. If G contains a closed Eulerian trail, it is **Eulerian**

Theorem 1. A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.

With this, we can easily characterize graphs possessing an Eulerian trail.

Corollary 2. A connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at another.

There is an analog to Eulerian trails. A path in a graph G that contains every vertex of G is called a **Hamiltonian path**.[1] Unfortunately, these are much less well-behaved.

2 Banquet Arrangement

Here's Problem 2 from the discussion worksheet.

Problem 1. Suppose n people are attending a banquet, and each of them has at least m friends ($2 \leq m \leq n$), where friendship is mutual. Prove that we can put at least $m+1$ of the attendants on the same round table, so that each person sits next to his or her friends on both sides.

Solution. As an example, suppose Alice, Bob, Claire, and Dom are attending, and the friend pairings are (Alice, Claire), (Alice, Dom), (Bob, Claire), (Bob, Dom), so $n = 4$ and $m = 2$

[1] We studied Eulerian *trails* because such trails may need to have repeated vertices, unlike paths which necessarily have unique vertices.

in this case. We actually can't choose exactly three of them so that they all sit next to their friends, but we can certainly choose to sit the four them in the order (Alice, Claire, Bob, Dom).

If you haven't realized already, it's helpful to recast this problem in terms of graphs to solve it in general. A big tipoff is both the fact that there are people and mutual friendships which strongly hint to being vertices and edges respectively as well as the topic of this discussion. So we will set

$$\text{people} \mapsto \text{vertices}$$

and

$$\text{friendship} \mapsto \text{edges}\ .$$

If we were to represent the above problem in graphs, it'd look something like this:

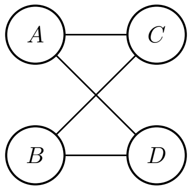

Figure 2: Casting the problem in terms of graphs.

We also see that sitting people around a table so that they sit next to their friends is just asking to find a cycle of length at least $m+1$. Using the above graph, it's easy to find one such cycle, $ACBD$, by starting at A and walking on the graph until you come back to A.

So that's the easy part of the problem. Now let's see how we would create such a cycle. Naively, we might start at an arbitrary vertex, say v_0, and try to walk along this graph. We know v_0 has at least m neighbors, so let's visit one of them, v_1.

Now v_1 also has at least m neighbors, but we've already been to one of them, v_0, so it really only has $m-1$ unvisited neighbors, so let's visit one of them, v_2. We can keep repeating this process of visiting unvisited neighbors until we're out of new neighbors to get a path

$$P = v_0 v_1 \dots v_l.$$

One thing we should note is that in the worst case, the number of unvisited neighbors for each v_i is at least $m-i$, which is if all of the i vertices already in the path are neighbors of v_i. So $l \geq m$, or else we would still have neighbors we could visit.

All that's left for us now is to get a cycle from this path. Since we're stuck when we reach v_l, all of its neighbors must be in P. So let's make a new path starting at v_l by backpedaling from v_l until we reach the last neighbor of v_l in P, say v_k. Now we have a path

$$P' = v_l v_{l-1} \dots v_k$$

where P' is also at least $m+1$ since v_l and all of its neighbors are in the path. But v_k is neighbors with v_l, so this is also a cycle, and we're done. □

We used the principle of trying things until we get stuck, and trying to fix things until we're free. You'll find that this often works surprisingly well.

The official solution is quite slick as well, but less motivated. It considers the *longest* path in the graph, and constructs the cycle from that. This is an example of the **Extremal Principle**, where one looks at either the largest or the smallest object satisfying some property.

Official Solution. Let $P = v_0 v_1 \dots v_l$ be a longest path in the graph. Such a path exists because the length of paths is bounded above by n. All neighbors of v_0 must be in P, since otherwise P can be extended to be even longer by appending this edge at the beginning of path P. Let k

be the maximum index of neighbors of v_0 along P. Since v_0 has at least m neighbors, we must have $k \geq m$. Then $v_0 v_1 \dots v_k v_0$ gives us the desired cycle. □

Here's a cute example of the Extremal Principle applied in another setting. Before you read on, think about it for a few minutes.

Problem 2. There are n students standing in a field such that the distance between each pair is distinct. Each student is holding a ball, and when the teacher blows a whistle, each student throws their ball to the nearest student. Prove that there is a pair of students that throw their balls to each other.

Proof. Consider the two students closest to each other. Since they are each other's nearest student, they must throw their balls to each other. □

Try this one for another extremal flavored graph problem. How would you do it without looking at the extreme cases?

Exercise 2.1. Show that every tree G contians at least two leaves, or vertices of degree 1.