Lagrange Interpolation

Tyler Zhu

October 25, 2018

The contents of this write-up are adapted from an old source of mine. Credits to that source.

1 Lagrange Interpolation

We revisit the problem of determining a polynomial $P(x)$ of degree *at most* n when we are given $n+1$ points,[1] but this time we attack it in a general, systematic matter that leads to Lagrange's beautiful interpolation formula.

Example 1. Let $P(x)$ be a cubic polynomial where $P(0) = 5$, $P(1) = 2$, $P(3) = 0$, and $P(4) = 6$. Find $P(2)$.

Solution. Instead of solving $ax^3 + bx^2 + cx + d = 0$ for a, b, c, d, we do something smarter.

Let's tackle the easier question of solving for a polynomial $p_1(x)$ such that $p_1(0) = 1$, $p_1(1) = 0, p_1(3) = 0$, and $p_1(4) = 0$. By the Factor Theorem, we know that $(x-1)(x-3)(x-4)$ divides $p_1(x)$. We simple need to multiply this by some constant c so that $p_1(0) = 1$. Hence,

$$p_1(x) = c(x-1)(x-3)(x-4) \implies 1 = c(0-1)(0-3)(0-4) \implies c = \frac{1}{(0-1)(0-3)(0-4)},$$

and so

$$p_1(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}.$$

Similarly, to find a polynomial $p_2(x)$ such that $p_2(0) = 0, p_2(1) = 1, p_2(3) = 0, p_2(4) = 0$, we can take

$$p_2(x) = \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}.$$

In the same way, let

$$p_3(x) = \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)},$$

$$p_4(x) = \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}.$$

Using a combination of these four polynomials, we can match any values at $x = 0, 1, 3$, and 4. For example, if we set

$$P(x) = y_1p_1(x) + y_2p_2(x) + y_3p_3(x) + y_4p_4(x),$$

then

$$\begin{aligned} P(0) &= y_1p_1(0) + y_2p_2(0) + y_3p_3(0) + y_4p_4(0) \\ &= y_1 \cdot 1 + y_2 \cdot 0 + y_3 \cdot 0 + y_4 \cdot 0 \\ &= y_1 \end{aligned}$$

[1] Why at most?

Similarly, $P(1) = y_2$, $P(3) = y_3$, and $P(4) = y_4$. All we have to do now is set our y_i to the given values to solve the problem. This gives us

$$\begin{aligned} P(x) &= 5 \cdot \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} + 2 \cdot \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)} \\ &\quad 0 \cdot \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)} + 6 \cdot \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} \\ &= \frac{5x^3 - 12x^2 - 29x + 60}{12} \end{aligned}$$

□

We can now write down the general formula.

Theorem 1 (Lagrange Interpolation Formula)**.** Let $(x_1, y_1), (x_2, y_2), \ldots, (x_{n+1}, y_{n+1})$ be $n+1$ distinct points. Then the polynomial defined by

$$P(x) = \sum_{i=1}^{n+1} y_i \frac{(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)(x-x_{n+1})}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)(x_i-x_{n+1})}$$

satisfies $P(x_i) = y_i$ for $i = 1, \ldots, n+1$.

Note that this interpolation is not entirely unique! If we had $P(1) = 1, P(3) = 9, P(6) = 36, P(7) = 49$, then obviously $P(x) = x^2$ works. But so does

$$P(x) = x^2 + (x-1)(x-3)(x-6)(x-7)(\pi x^4 + ex^3 - \sqrt{5}x + 1).$$

In fact, any polynomial of the form

$$P(x) = x^2 + (x-1)(x-3)(x-6)(x-7)q(x)$$

works, where $q(x)$ is a polynomial. Our intuition tells us that $P(x) = x^2$ should be the "best" solution to the interpolation; how do we make this precise?

Note that if $q(x)$ is a nonzero polynomial, then $P(x)$ becomes a polynomial of degree at least 4. Hence, $P(x) = x^2$ is the unique solution that has degree at most 3 (remember, $n+1$ points means degree n). So the Interpolation Formula does give us a unique solution; it is simply up to degree n.