Hard Counting Problems

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April 6, 2020

These are some pretty arbitrary, hard counting problems. They are more so for testing your general ability to see connections in counting well than how well you bookkeep. Safe travels.

1 Warm up

Problem 1. How many cubic polynomials $f(x)$ with positive integer coefficients are there such that $f(1) = 9$?

2 Assorted Candies

Problem 2. Prove $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$ with a combinatorial argument.

Problem 3. Let $a_1, a_2, \ldots, a_n$ be a sequence of arbitrary natural numbers. Define b_k to be the number of elements a_i for which $a_i \geq k$. Prove that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots$.

Problem 4. How many ways are there to insert +'s between the digits of 111111111111111 (fifteen 1's) so that the result will be a multiple of 30?

Problem 5. Compute

$$\sum_{n_{60}=0}^{2} \sum_{n_{59}=0}^{n_{60}} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} \sum_{n_0=0}^{n_1} 1.$$

Problem 6. There's a new (virtual) game show featuring N people where a few lucky contestants get to compete for the ultimate prize: a roll of toilet paper. The game works as follows: everyone lines up in front of a jar of ping pong balls numbered 1 through 100 and one-by-one randomly select a ball until everyone has one. Then the winning number is announced, and anyone with the winning number wins the prize.

(a) Suppose the winning number is 42. What's the probability that if $N = 3$, then the third person wins the game?

(b) If $N = 100$, what's the probability that the third person wins the game now?

(c) You and a friend are watching the game (for $N = 100$) and after all of the balls have been drawn, you both decide to bet on the results. Your friend picks contestant 42, thinking they must have the winning number, but before you can pick a contestant, 98 groan in realization that they have losing numbers, which leaves you with no choice but to bet on contestant 20. What's the probability that you win the bet?

3 Dessert

These aren't actually relevant to course material; please don't do them unless you really want to!

Exercise 3.1. Let $(a_1, a_2, \ldots, a_{12})$ be a permutation of $(1, 2, \ldots, 12)$ for which

$$a_1 > a_2 > a_3 > a_4 > a_5 > a_6 \text{ and } a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}.$$

An example of such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$. Find the number of such permutations.

Exercise 3.2. Let N denote the number of 7-tuples of sets $S_1, S_2, \ldots, S_7$, not necessarily distinct, for which

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_7 \subseteq \{1, 2, 3, 4, 5, 6, 7\}.$$

Find N.

Exercise 3.3. In a shooting match a marksman must break eight targets arranged in three hanging columns of 3, 3 and 2 targets respectively. Whenever a target is broken, it must be the lowest unbroken target in its column. In how many different orders can the eight targets be broken?