PÓLYA'S RECURRENCE THEOREM

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1. Introduction

In this article, we discuss the classic Pólya's Recurrence Theorem about random walks on $\mathbb{Z}^d$, the integer lattice in d dimensions, and prove the result in full generality using elementary techniques aided by Stirling's Approximation.

2. Pólya's Recurrence Theorem

"A drunk man will find his way home, but a drunk bird may get lost forever."

– *Shizuo Kakutani.*

Most simply stated, a random walk is simply a path on a graph or lattice where each step is determined randomly according to some probability distribution, such as a coin or a fair die. Each outcome of flipping a coin, heads or tails, would corresponding to a certain move, such as moving left or right on a number line, or moving up or down in $\mathbb{Z}^2$.

A random walk which returns to its starting position after a finite number of steps is said to be *recurrent*. Any random walk that is not recurrent is *transient*. The Hungarian mathematician George Pólya came up with a characterization of all random walks on $\mathbb{Z}^d$ in these terms.

Theorem 1. *The simple random walk on $\mathbb{Z}^d$ is recurrent in dimensions $d = 1, 2$ and transient for dimensions $d \geq 3$.*

A classic application of the theorem is the quote mentioned at the beginning of this section. A drunk man has a probability of 1 of returning to his home since he moves in $\mathbb{Z}^2$, but a drunk bird is not as lucky since it moves in $\mathbb{Z}^3$, and thus has a positive probability of never returning to its home. On the note of real world applications, another interesting example is that wind instruments can exist in a world like ours, but not in one such as Flatland, which is 2D. See [3] for a proof of the theorem as a formulation in electrical currents, from which a different intuition is gained and perhaps a real world application as well.

We will begin by first introducing some preliminary notation suggested by [1] that will be useful in proving the theorem, and then do so directly for $d = 1, 2$. Then, we will show that when $d = 3$, the simple random walk is transient, which in turn implies that it is transient in dimensions $d \geq 3$.

3. Preliminaries

Let

$$u_n = \text{ the probability that a random walk is at 0 after } n \text{ steps}$$

for $n \geq 0$. Then it is clear that $u_0 = 1$, and that $u_1 = u_3 = u_5 = \cdots = 0$, since you must be at an even number of steps to be at 0. Let

$$f_n = \text{ the probability that a random walk returns to 0 for the first time at step } n$$

for $n \geq 1$, where we let $f_0 = 0$ since you don't really "return" to 0 at the beginning. Also, let $f = \sum_{n=0}^{\infty} f_n$. Then, we see that

$$\text{Probability of the random walk returning to 0 } = f.$$

$$\text{Probability of the random walk not returning to 0 } = 1 - f.$$

Now, we will decompose each of the u_n by considering the different times for when the random walk can first return to 0, a.k.a. f_i. Consider the following equations:

$$\begin{aligned} u_1 &= f_0 u_1 + f_1 u_0 \\ u_2 &= f_0 u_2 + f_1 u_1 + f_2 u_0 \\ &\vdots \\ u_n &= f_0 u_n + f_1 u_{n-1} + \cdots + f_n u_0 \end{aligned}$$

Take u_2 as an example. There are "three" ways we can be at 0 after 2 steps:

(1) Returning to 0 for the first time at step 0, which by definition has probability 0
(2) Returning for the first time at step 1, which means you need go back to 0 after 1 step
(3) Returning for the first time at step 2, after which you stay there since you've already used two steps.

In general, each decomposition is modeled by $f_k u_{n-k}$, where the random walk returns for the first time after k steps and then in $n - k$ steps is back at 0, and repeating the above argument will give the equalities. Now, let

$$\begin{aligned} U(s) &= u_0 + u_1 s + u_2 s^2 + \cdots \\ F(s) &= f_0 + f_1 s + f_2 s^2 + \cdots \end{aligned}$$

Then,

$$\begin{aligned} U(s)F(s) = u_0 f_0 + (f_0 u_1 + f_1 u_0)s + (f_0 u_2 + f_1 u_1 + f_2 u_0)s^2 + \cdots = U(s) - 1 \\ \implies 1 = U(s)(1 - F(s)). \end{aligned}$$

Now, we will need the following lemma that gives us a simple criteria for recurrence and transience.

Lemma 2. *If* $\sum_{n=0}^{\infty} u_n = \infty$*, then* $f = 1 \implies$ *the random walk is recurrent.*

If $\sum_{n=0}^{\infty} u_n < \infty$*, then* $f < 1 \implies$ *the random walk is transient.*

Proof. $\sum_{n=0}^{\infty} u_n = \infty$ tells us that $\lim_{s \to 1} U(s) = \infty$, and so

$$\lim_{s \to 1} 1 - F(s) = \lim_{s \to 1} \frac{1}{U(s)} = 0 \implies \lim_{s \to 1} F(s) = 1 \implies f = 1,$$

which says that the random walk is recurrent. $\sum_{n=0}^{\infty} u_n < \infty$ tells us that $\lim_{s\to 1} U(s) < \infty$, and so

$$\lim_{s\to 1} 1 - F(s) = \lim_{s\to 1} \frac{1}{U(s)} > 0 \implies \lim_{s\to 1} F(s) < 1 \implies f < 1,$$

which says that the random walk is transient. $\square$

With this lemma, we can now calculate the sum of the u_n's to determine the characterization of a random walk. Thus, the problem reduces to calculating u_n for each dimension and then summing from there.

4. Random Walks on the Number Line

We will first consider random walks on the number line, or $\mathbb{Z}^1$. The first simplification we can make is that for n odd, $u_n = 0$. Thus, we know that $\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} u_{2n}$. Calculating u_{2n} is a classical combinatorial problem. In order to end up at 0, there must be an equal number of left moves as right moves, and so the problem reduces to rearranging n L's and n R's, which has $\binom{2n}{n}$ ways of being done. Dividing this by the total number of possibilities, 2^{2n}, gives that

$$u_{2n} = \frac{\binom{2n}{n}}{2^{2n}}.$$

Now, we can use the following well known approximation for the factorial to estimate the value of u_{2n}.

Lemma 3 (Stirling's Formula)**.** $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Corollary 4. $\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}$.

Proof. By Stirling's Formula, we see that

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \sim \frac{\sqrt{2\pi \cdot 2n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2} = \frac{\sqrt{2}\cdot\sqrt{2\pi n}\cdot 2^{2n}\cdot\left(\frac{n}{e}\right)^{2n}}{(\sqrt{2\pi n})^2\left(\frac{n}{e}\right)^{2n}} = \frac{2^{2n}}{\sqrt{\pi n}}.$$

$\square$

Putting this all together, we see that

$$\sum_{n=0}^{\infty} u_{2n} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \sim \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty$$

by standard analytic techniques. Hence, by Lemma 2, the random walk is recurrent.

5. Random Walks on the Number Lattice

Now we consider random walks on $\mathbb{Z}^2$. We will use a nice trick that makes this calculation much simpler than it would be otherwise. Rotate $\mathbb{Z}^2$ about the origin by 45°, and then dilate by $\sqrt{2}$. Effectively what we are doing is turning our lattice into a diagonal lattice where every move is either $(+1,+1),(+1,-1),(-1,+1),(-1,-1)$. Considering the x and y components separately, it is straightforward to see that each one can be thought of as a random walk on $\mathbb{Z}^1$ itself, and thus

$$u_{2n} = \left(\frac{1}{\sqrt{\pi n}}\right)^2 = \frac{1}{\pi n}.$$

Hence,

$$\sum_{n=0}^{\infty} u_{2n} = \sum_{n=0}^{\infty} \frac{1}{\pi n} = \infty,$$

from which it follows that this random walk is also recurrent.

6. Random Walks on the Number Cube

To cap this all off, we will now calculate u_{2n} for random walks on $\mathbb{Z}^3$ with a method suggested by [2]. We know that there are 6 possible directions to move in, one negative and one positive direction for each of the three axes, so there are a total of 6^{2n} different possible random walks. Of these walks, we are interested in the ones that return to 0. Let

$$\begin{aligned} j &= \text{ the number of steps in direction } (1,0,0) \\ k &= \text{ the number of steps in the direction } (0,1,0) \end{aligned}$$

In order to return to 0 after the $2n$ steps, there must be an even number of moves in each axis direction, and an equal number in the positive and negative directions for each axis direction. Thus, there are $n-j-k$ steps in direction $(0,0,1)$. The other n steps will be in the respective opposite directions. Now, this just amounts to finding the number of ways to rearrange $2n$ different letters with repeats of $j,j,k,k,n-j-k,n-j-k$. Summing this over all possible j and k gives that

$$\begin{aligned} u_{2n} &= 6^{-2n} \sum_{\substack{j,k\geq 0 \\ j+k\leq n}} \frac{(2n)!}{(j!\cdot k!\cdot (n-j-k)!)^2} \\ &= 6^{-2n}\binom{2n}{n} \sum_{\substack{j,k\geq 0 \\ j+k\leq n}} \left(\frac{n!}{j!\cdot k!\cdot (n-j-k)!}\right)^2 \\ &= 2^{-2n}\binom{2n}{n} \sum_{\substack{j,k\geq 0 \\ j+k\leq n}} \left(3^{-n}\frac{n!}{j!\cdot k!\cdot (n-j-k)!}\right)^2 \end{aligned}$$

The easy part is done now, and what is left is to asymptotically estimate this. Now observe that

$$\sum_{\substack{j,k\geq 0 \\ j+k\leq n}} 3^{-n}\frac{n!}{j!\cdot k!\cdot (n-j-k)!} = 1$$

because there is a bijection between the number of rearrangements of j A's, k B's, and $n-j-k$ C's and the number of n letter strings of A's, B's, or C's. There are 3^n such n letter strings, from which we conclude our observation. Now it is possible to show that

$$\sum_{\substack{j,k\geq 0 \\ j+k\leq n}} \left(3^{-n}\frac{n!}{j!\cdot k!\cdot(n-j-k)!}\right)^2 \leq \max\left(3^{-n}\frac{n!}{j!\cdot k!\cdot(n-j-k)!}\right),$$

from which it follows that

$$u_{2n} \leq 2^{-2n}\binom{2n}{n}\max\left(3^{-n}\frac{n!}{j!\cdot k!\cdot(n-j-k)!}\right).$$

Using Stirling's Formula again, we see that

$$\begin{aligned}\frac{n!}{j!\cdot k!\cdot(n-j-k)!} &\sim \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi j}\left(\frac{j}{e}\right)^j\sqrt{2\pi k}\left(\frac{k}{e}\right)^k\sqrt{2\pi(n-j-k)}\left(\frac{n-j-k}{e}\right)^{n-j-k}} \\ &= \frac{\sqrt{n}}{\sqrt{j\cdot k\cdot(n-j-k)}}\frac{n^n}{j^jk^k(n-j-k)^{n-j-k}}.\end{aligned}$$

It is a standard exercise in analysis to see that $j=k=n-j-k\approx\frac{n}{3}$ maximizes this value. For ease of computation, assume that $\frac{n}{3}$ is integer; we can do this as we are mainly interested in the asymptotics of the value. Thus, we get that

$$\begin{aligned}\max\left(3^{-n}\frac{n!}{j!\cdot k!\cdot(n-j-k)!}\right) &= 3^{-n}\cdot\frac{\sqrt{n}}{\sqrt{\frac{n}{3}\cdot\frac{n}{3}\cdot\frac{n}{3}}}\cdot\frac{n^n}{\left(\frac{n}{3}\right)^{n/3}\left(\frac{n}{3}\right)^{n/3}\left(\frac{n}{3}\right)^{n/3}} \\ &= 3^{-n}\cdot\frac{c\sqrt{n}}{n\sqrt{n}}\cdot\frac{n^n}{\left(\frac{n}{3}\right)^n} \\ &= 3^{-n}\cdot\frac{c}{n}\cdot 3^n = \frac{c}{n}\end{aligned}$$

where c is probably $3\sqrt{3}$, but for our purpose is simply some constant. Combining all of this, we see that

$$u_{2n} \leq 2^{-2n}\binom{2n}{n}\cdot\frac{c}{n} \sim \frac{1}{\sqrt{\pi n}}\cdot\frac{c}{n} = \frac{c}{n^{3/2}} \implies \sum_{n=0}^{\infty} u_{2n} = \sum_{n=0}^{\infty}\frac{c}{n^{3/2}} < \infty,$$

and hence by Lemma 2, the random walk in $\mathbb{Z}^3$ is transient.

7. Random Walks in Higher Dimensions and Decimal Dimesions

Note that any random walk in the 4th dimension and higher can be represented as a tuple of at least 3 integer coordinates. However, since we already showed that random walks on the number cube are transient, no matter what higher dimension we randomly walk in, the first three coordinates will never return back to 0 at the same time, aRnd so neither will all of the coordinates. Thus, all higher random walks are transient.

It is of interest to note that if we used slightly stronger arguments to show that the random walk on $\mathbb{Z}^n$ is transient for all real numbers $n>2$, then we could ask ourselves what a random walk on $\mathbb{Z}^{2.1}$ might mean. One way is to think of this as a random walk on $\mathbb{Z}^3$ with coordinates (x,y,z), except with certain restrictions on movement in the z direction.

For example, making $z = f(x, y)$ where f is a function of x and y makes the random walk simply a random walk on $\mathbb{Z}^2$. However, if we have a different constraint such as $z \leq x + y$, or $x^2 + y^2 + z^2 \leq r^2$, then a random walk subject to these conditions will be transient.

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References

[1] Charles N. Moore. *Random Walks*, available at `https://www.math.ksu.edu/c̃nmoore/randomwalk.pdf`.

[2] Ron Peled. *Random Walks and Brownian Motion: Lecture 4*, available at `http://www.math.tau.ac.il/p̃eledron/Teaching/RW_and_BM_2011/scribe4.pdf`.

[3] P. G. Doyle and J. L. Snell. *Random Walks and Electric Networks*. The Mathematical Association of America, 1984.