Poles and Polars

Irvington Math Club

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Abstract

The goal of this handout is to provide the reader with some of the intuition and common techniques for solving problems by poles and polars. I would like to thank Alexander Remorov and Yufei Zhao for publishing some amazing projective geometry handouts that helped in making this handout, as well as Yiwen Dai for providing helpful feedback.

In short, poles and polars are usually involved in problems asking you to prove collinearity, perpendicularity, and that's about it. Cool, let's get started!

1 Poles and Polars: Definitions

Suppose that ω is a circle with center O. Then for any point P in the same plane as ω, we can define something called the **inverse** of point P **with respect to (w.r.t.)** circle ω.

Definition 1.1 (The inversive transformation)**.** If ω is a circle with center O and radius r, and P is a point $\neq O$, then we define the **inverse** of point P, P' with respect to circle ω to be the point on the ray $\overrightarrow{OP}$ such that

$$\overline{OP} \cdot \overline{OP'} = r^2.$$

Next, we define **poles and polars**.

Definition 1.2 (Poles and Polars)**.** We define the **polar** of point P w.r.t. ω to be the line p perpendicular to the ray $\overrightarrow{OP}$ at P', the inverse of P w.r.t. ω. P is called the **pole** of line p w.r.t. ω.

In the following diagram, P would be the pole of line p with respect to ω.

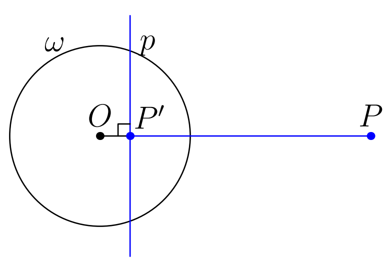


Figure 1: P is the pole of p, and p is the polar of P.

Basically, If you want to prove something about the polar of a point, it suffices to prove it for its pole. There will be examples of this later on.

I stress: **Poles are POINTS, Polars are LINES**.

I repeat! **Poles are POINTS, Polars are LINES**

2 Poles and Polars: Configurations

Now, we must establish some well known configurations regarding poles and polars. Know these well as they will come up again and again.

Lemma 2.1. *If P lies outside of circle ω, then the polar of P w.r.t. ω is the line determined by the tangency points of the tangents from P to ω. In other words, if the tangents from P to ω are $\overline{PE}$ and $\overline{PF}$, then the polar of P would be $\overline{EF}$.*

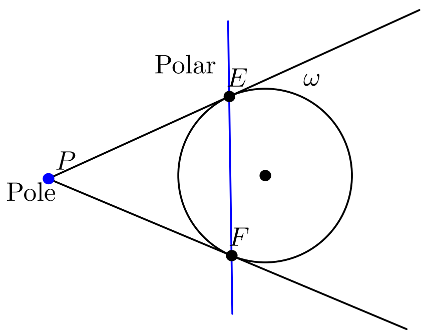


Figure 2: Lemma 2.1

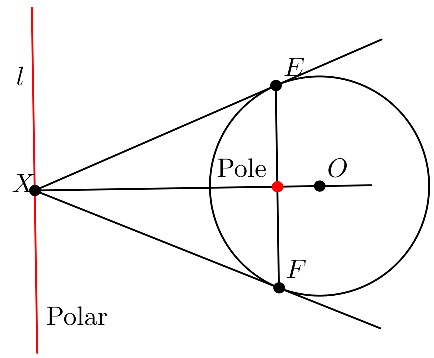


Figure 3: Lemma 2.2

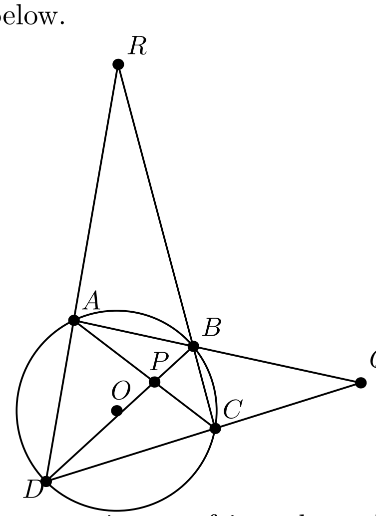
Lemma 2.2. *If a line l lies outside of circle ω with center O, then the pole of line l w.r.t. ω can be found as follows. Construct a perpendicular from the center of ω to line l, and call the foot of the perpendicular X. Then, the pole of l is the intersection point of the polar of X and the ray $\overrightarrow{OX}$.*

3 Poles and Polars: Theorems

Theorem 1 (La Hire's). *If a point X lies on the polar of point Y with respect to ω, then the point Y lies on the polar of point X with respect to ω.*

This theorem is more of a helping "tool" than a theorem. It often lets us compare two points or lines to a common polar or pole, which will usually give us our desired concurrency or perpendicularity. If this doesn't make sense yet, then read the examples below.

Theorem 2 (Brokard's with a K). *The points A, B, C, D lie in this order on circle ω with center O. $\overline{AC}$ and $\overline{BD}$ intersect at P, $\overline{AB}$ and $\overline{DC}$ intersect at Q, $\overline{AD}$ and $\overline{BC}$ intersect at R. Then O is the orthocenter[a] of $\triangle PQR$. Furthermore, $\overline{QR}$ is the polar of P, $\overline{PQ}$ is the polar of R, and $\overline{PR}$ is the polar of Q w.r.t. ω.*

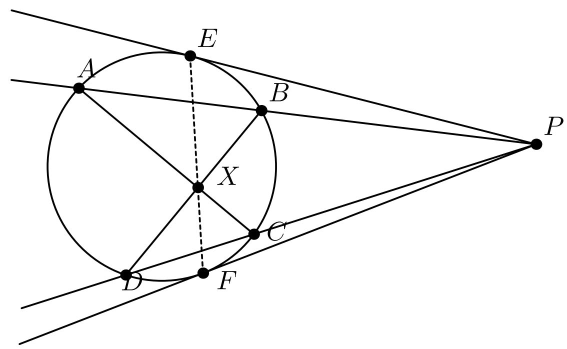


[a] The orthocenter of triangle ABC is the intersection of the altitudes of the triangle.

This theorem is a powerful one! Make sure you know this by heart, as its proof is rather advanced. Most projective geometry problems come down to finding the correct configurations to apply Brokard's to, so make sure to "complete" the diagram if you are ever stuck! See the examples below for what I mean by completing the diagram.

4 Trivial Examples

***Example* 4.1:** Let A, B, C, D be points on a circle in that order, and let P be the intersection of $\overline{AB}$ and $\overline{CD}$. Let $\overline{PE}$ and $\overline{PF}$ be tangents to the circle. Prove $\overline{AC}$, $\overline{BD}$, $\overline{EF}$ are concurrent.

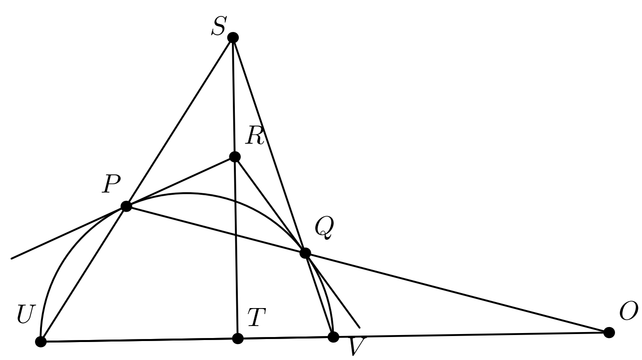


Proof. We first let the intersection of $\overline{AC}$ and $\overline{BD}$ be X, so that we want to prove that X lies on $\overline{EF}$. Since P is the intersection of the tangents at E and F, $\overline{EF}$ is the polar of P. Now at this point, there isn't much information left for us to find, so we decide to **complete the diagram.**

Seeing that the circle and the lines $\overline{AP}, \overline{DP}$ remind us of Brokard's Theorem, we complete the diagram by extending $\overline{AD}$ and $\overline{BC}$ to meet at Q. By Brokard's, $\overline{XQ}$ is the polar of P. However, we already found that $\overline{EF}$ is the polar of P, which means that $\overline{EF}$ and $\overline{XQ}$ are the same line. Therefore, X must lie on $\overline{EF}$, finishing our proof. □

One-Liner. X lies on the polar of P by Brokards, but EF is the polar of P, implying that it must go through X. □

***Example* 4.2:** Let $\overline{UV}$ be a diameter of a semicircle, and let P, Q are two points on the semicircle. The tangents to the semicircle at P and Q meet at R, and lines $\overline{UP}$ and $\overline{VQ}$ meet at S. Prove that $\overline{RS} \perp \overline{UV}$.



Proof. Seeing that R is the intersection of the tangents immediately makes us think of Lemma 2.1, which tells us that $\overline{PQ}$ is the polar of R, but there isn't much else. To try and gain more information, we, again, **complete the diagram** by extending $\overline{PQ}$ to meet $\overline{UV}$ at O. Aha! We can now apply Brokard's to our diagram to see that $\overline{PO}$ is the polar of S. Now, we want to prove that $\overline{RS}$ is tangent to $\overline{UV}$, and since $\overline{UV}$ is the same line as $\overline{UO}$, we should somehow involve the polar of O since that is perpendicular to $\overline{UO}$. Wait a second...

By La Hire's, we see that since O lies on the polar of R (O lies on $\overline{PQ}$), then R lies on the polar of O. Similarly, S lies on the polar of O. However, $\overline{RS}$ is the same line as said polar, yet the polar is also perpendicular to $\overline{UV}$. Therefore, $\overline{RS} \perp \overline{UV}$, concluding our proof. □

5 Trivial Problems

***Problem* 5.1:** Let ABC be a triangle and I be its incenter. Let the incenter of ABC touch sides $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F respectively. Let S denote the intersection of lines $\overline{EF}$ and $\overline{BC}$. Prove that $\overline{SI} \perp \overline{AD}$.

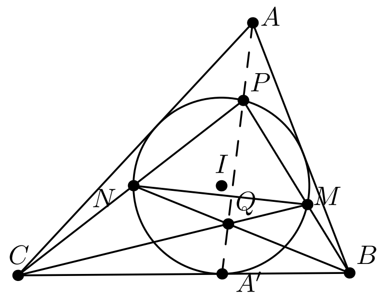
***Problem* 5.2:** A circle is inscribed in quadrilateral $ABCD$ so that it touches sides $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ at E, F, G, H respectively.

(a) Show that lines $\overline{AC}, \overline{EF}, \overline{GH}$ are concurrent. In fact, they concur at the pole of $\overline{BD}$.

(b) Show that lines $\overline{AC}, \overline{BD}, \overline{EG}, \overline{FH}$ are concurrent.

***Problem* 5.3** (China 1996)**:** Let H be the orthocenter of triangle ABC. From A construct tangents $\overline{AP}$ and $\overline{AQ}$ to the circle with diameter $\overline{BC}$, where P, Q are the points of tangency. Prove that P, H, Q are collinear.

***Problem* 5.4:** Let ABC be a triangle. The incircle touches the side $\overline{BC}$ at A' and the line $\overline{AA'}$ meets the incircle again at a point P. Let the lines $\overline{CP}$ and $\overline{BP}$ meet the incircle of triangle ABC again at N and M respectively. Prove that the lines $\overline{AA'}, \overline{BN}$, and $\overline{CM}$ are concurrent.[1]



6 Nontrivial Problems

***Problem* 6.1** (IMO 1985)**:** A circle with center O passes through the vertices A and C of triangle ABC and intersects segments $\overline{AB}$ and $\overline{BC}$ again at distinct points K and N, respectively. The circumcircles of triangles ABC and KBN intersect at exactly two distinct points B and M. Prove that $\angle OMB = 90^\circ$.

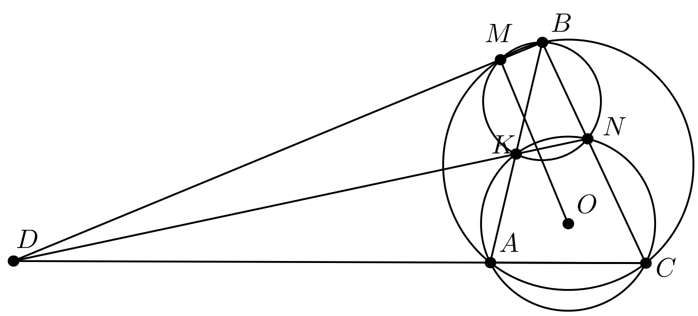


Figure 4: Problem 6.1

[1]Hint: Let Q be the intersection of $\overline{BN}$ and $\overline{CM}$. We want to prove that Q lies on $\overline{AA'}$. Look familiar?