MODULI SPACES

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Abstract. In this paper, we discuss the notion of a moduli space and determine moduli spaces of isomorphism classes of genus 0 and 1 curves with various numbers of marked points.

1. Moduli Spaces

1.1. **Introduction.** According to [2], moduli spaces can be thought of as geometric solutions to geometric classification. In broad terms, a moduli problem consists of three problems:

(1) Objects: which geometric objects would we like to describe, or *parametrize*?
(2) Equivalences: when do we identify two of our objects as being isomorphic, or being "the same"?
(3) Families: how do we allow our objects to vary, or modulate?

We will demonstrate these ideas with the following problem:

Problem 1.1: Describe the collections of lines that pass through the origin in $\mathbb{R}^2$.

From here on out, lines will be understood to pass through the origin in $\mathbb{R}^2$. We can easily solve this classification problem by assigning each line L a parameter $\theta \in [0, \pi)$ that represents its angle with the x-axis. Hence, this set of lines, the real projective line known as $\mathbb{RP}^1$, is in one-to-one correspondence with the half open interval $[0, \pi)$.

However, this geometric solution doesn't capture the natural topology of the problem. It doesn't demonstrate how the line L with $\theta(L) = 0$ is nearby lines with $\theta = 0.1$, but also with $\theta = \pi - 0.1$. One way of fixing this is to instead consider the closed interval $[0, \pi]$ and identify 0 and π to be the same point. Formally, we would be considering

$$[0, \pi]/\sim \qquad \text{where } 0 \sim \pi.$$

Now, numbers close to 0 are also close to π, and vice versa. In other words, we glued together the ends of a line segment to form a circle. See Figure 1

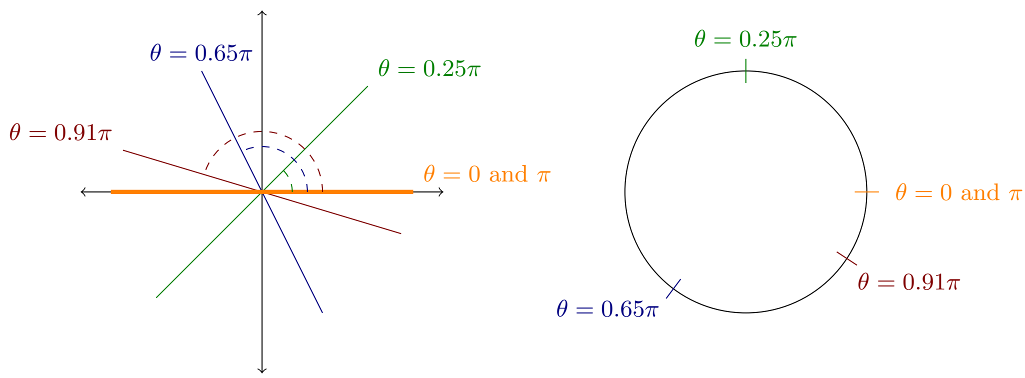

Figure 1. An example of the correspondence $[0, \pi]/\sim$

Perhaps a more natural way to arrive at the same construction is to consider the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. For each $s \in \mathbb{S}^1$, we can consider the line $L(s)$ passing through the origin and s. Now, we have a family of lines parametrized by $\mathbb{S}^1$, that is, a map $s \mapsto L(s)$ that takes points on $\mathbb{S}^1$ to lines in $\mathbb{RP}^1$. However, this map is two-to-one since s and $-s$ map to the same line. We fix this by identifying s and $-s$ to be the same and letting the s with angles in $[0, \pi)$ be representatives, giving us a one-to-one correspondence between $\mathbb{RP}^1$ and something that is still topologically equivalent to a circle. See Figure 2.

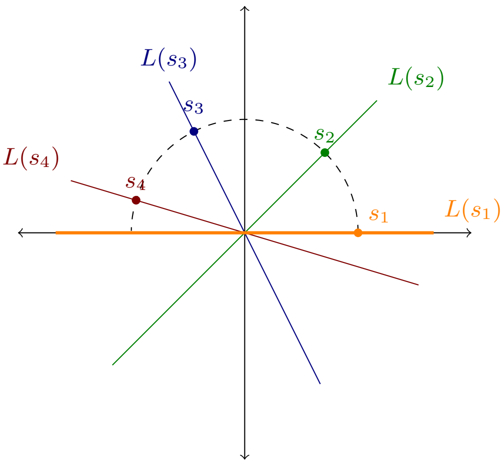

FIGURE 2. An example of the correspondence $s \mapsto L(s)$

Our task is similar to these solutions. We wish to find some space or variety that parametrizes each object in a given family such that this space preserves any inherent topological notions.

1.2. **Objective.** The main objects of our study are smooth compact complex curves, also called Riemann surfaces, with n marked numbered pairwise distinct points. Unless otherwise specified they are assumed to be connected. Every compact complex curve has an underlying structure of a 2- dimensional oriented smooth compact surface, that is uniquely characterized by its genus g.

One of the main goals of this paper is to understand the moduli space of various families of curves, which we define now.

Definition 1. The *moduli space* $\mathcal{M}_g$ is the set of isomorphism classes of curves of genus g.

In addition, for $2 - 2g - n < 0$, the *moduli space* $\mathcal{M}_{g,n}$ is the set of isomorphism classes of Riemann surfaces of genus g with n marked points, i.e. distinct ordered marked points on one curve must map to the corresponding marked point on the other curve.

2. RATIONAL CURVES

We wish to study the moduli space of genus 0 curves, $\mathcal{M}_{0,n}$ where $n > 2$. Curves of genus 0 are called *rational curves*, and they all turn out to be conics.

Let us first make some observations about rational curves. Since they are genus 0, when drawn over $\mathbb{C}^2$ (which cannot be visualized since it has 4 real dimensions), the curve is homeomorphic to a genus 0 surface, or a sphere, upon compactification. Such a sphere is also isomorphic to the complex projective line[1] $\mathbb{CP}^1$.

Now let us consider some basic properties of rational curves by studying $\mathcal{M}_{0,3}$. Our main reference here is [3].

Example 2.1: We claim that every rational curve (C, x_1, x_2, x_3) with three marked points is isomorphic to $(\mathbb{CP}^1, 0, 1, \infty)$ in a unique way. Hence, $\mathcal{M}_{0,3}$ = a point.

[1]A line with one complex dimension, or a plane with two real dimensions.

To study isomorphism classes of rational curves, we look to the automorphism group of $\mathbb{CP}^1$, which is $\mathrm{PSL}(2,\mathbb{C})$ acting by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Consider the group action

$$\begin{pmatrix} x_3 - x_2 & -x_1(x_3-x_2) \\ x_1 - x_2 & -x_3(x_1-x_2) \end{pmatrix}$$

which sends (C, x_1, x_2, x_3) to $(\mathbb{CP}^1, 0, 1, \infty)$. If such an action exists, this will be unique (up to scaling). One way to see this is to solve the system of equations

$$\frac{ax_1+b}{cx_1+d} = 0 \tag{1}$$

$$\frac{ax_2+b}{cx_2+d} = 1 \tag{2}$$

$$\frac{ax_3+b}{cx_3+d} = \infty \tag{3}$$

where equation (3) can be solved by considering when $cx_3 + d = 0$. Setting $c = 1$ and clearing denominators gives the action above and proves our claim.

Example 2.2: Now we will identify $\mathcal{M}_{0,4}$. This proof is taken from [3]. Every curve (C, x_1, x_2, x_3, x_4) is isomorphic to $(\mathbb{CP}^1, 0, 1, \infty, t)$ in a unique way. The number $t \neq 0, 1, \infty$ is determined by the positions of the marked points on C. It is called the *modulus* and gave rise to the term moduli space. If $C = \mathbb{CP}^1$, then t is the cross-ratio of x_1, x_2, x_3, x_4. The moduli space $\mathcal{M}_{0,4}$ is the set of values of t, that is $\mathcal{M}_{0,4} = \mathbb{CP}^1 \backslash \{0, 1, \infty\}$.

These two examples lead us to conjecture the following classification theorem for the moduli space of genus 0 curves with n marked points for $n > 0$.

Theorem 2. *The moduli space $\mathcal{M}_{0,n}$ is given by*

$$\mathcal{M}_{0,n} = \{(t_1, \ldots, t_{n-3}) \in (\mathbb{CP}^1)^{n-3} \mid t_i \neq 0, 1, \infty, \quad t_i \neq t_j\}.$$

The proof is similar, because the curve $(C, x_1, \ldots, x_n)$ can be uniquely identified with $(\mathbb{CP}^1, 0, 1, \infty, t_1, \ldots, t_{n-3})$.

3. Elliptic Curves

The main reference used for this section is [1].

Elliptic curves are curves of genus one with a specified base point. This means that if we formed a surface from the curve (over the complexes, which has thickness), we would have a genus 1 surface, or a torus. It is particularly simple to classify all elliptic curves up to isomorphism, and we do so algebraically by the Weierstrass Equation.

3.1. The Weierstrass Equations.

Every elliptic curve can be expressed as a curve in $\mathbb{P}^2$ with its one base point on the line at infinity. After scaling, we have that the equation is

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

where $O = [0 : 1 : 0]$ is the basepoint and $a_i \in \bar{K}$, an algebraically closed field. We call this general form of the elliptic curve a *Weierstrass Equation.*

To simplify this equation, we want to let $Z = 1$. Before we can, we have to check that there are no other solutions at infinity (i.e. making sure we don't divide by 0), which is true since $[0 : 1 : 0]$ is a triple point. Thus, dehomogenizing the equation and rewriting it gives

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Now, if char $\bar{K} \neq 2, 3$, then through a series of substitutions, we can arrive at the simpler equation

$$E : y^2 = x^3 + Ax + B$$

called the *short Weierstrass form.* The assumptions about the characteristic of $\bar{K}$ are necessary to complete the square and divide by 3. See Figure 3 for some examples of elliptic curves.

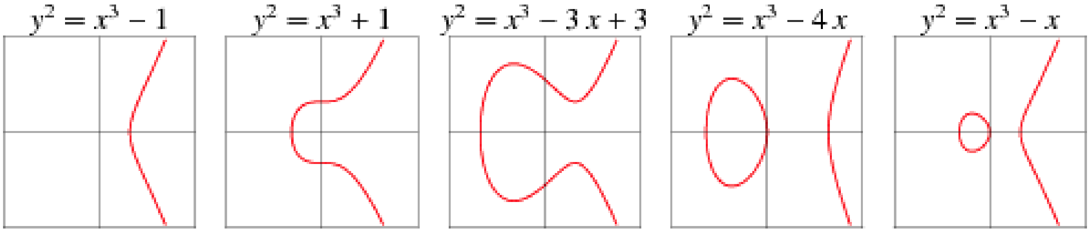

FIGURE 3. Five elliptic curves with various A and B.

3.2. **The j-invariant.** Now with this Weierstrass equation, we can associate the quantities

$$\Delta = -16(4A^3 + 27B^2) \qquad \text{and} \qquad j = -1728\frac{(4A)^3}{\Delta}.$$

We call Δ the *discriminant* and j the *j-invariant* of the elliptic curve, for reasons we will see shortly. We note that if $\Delta = 0$, then we define j to be ∞. Elliptic curves with $\Delta = 0$ are precisely those with singularities like cusps or nodes. The only change of variables which preserves this form of the equation is

$$x = u^2x' \qquad \text{and} \qquad y = u^3y' \qquad \text{for some } u \in \bar{K};$$

for which the new variables are

$$u^4A' = A, \qquad u^6B' = B, \qquad u^{12}\Delta' = \Delta.$$

This gives a criterion for when elliptic curves are isomorphic, which we can use to demonstrate the invariance of the so-called "j-invariant".

Proposition 3. *Two elliptic curves are isomorphic over $\mathbb{C}$ if and only if they both have the same j-invariant.*

Proof. If two elliptic curves are isomorphic, then the change of variables above shows us that

$$j' = -1728\frac{(4u^4A')^3}{u^{12}\Delta'} = -1728\frac{(4A')^3}{\Delta'},$$

so their j-invariants are the same.

For the converse, let E and E' be elliptic curves with the same j-invariant, say with short Weierstrass equations

$$E : y^2 = x^3 + Ax + B$$
$$E' : y^2 = x^3 + A'x + B'.$$

Then the assumption that $j(E) = j(E')$ gives

$$\frac{(4A)^3}{(4A^3 + 27B^2)} = \frac{(4A')^3}{(4A'^3 + 27B'^2)},$$

which yields

$$A^3B'^2 = A'^3B^2.$$

We look for an isomorphism of the form $(x, y) = (u^2x', u^3y')$ and consider three cases:

(1) $A = 0 (j = 0)$. Then $B \neq 0$, since $\Delta = 0$, so $A' = 0$. We obtain an isomorphism with $u = (B/B')^{1/6}$.
(2) $B = 0 (j = 1728)$. Then $A \neq 0$, so $B' = 0$. We pick $u = (A/A')^{1/4}$ for an isomorphism.

(3) $AB \neq 0 (j \neq 0, 1728)$. Then $A'B' \neq 0$, since if one were 0, both would be 0, contradicting $\Delta' \neq 0$. Taking $u = (A/A')^{1/4} = (B/B')^{1/6}$ gives our desired isomorphism.

□

The final step of the puzzle is now to determine the values for which there is an associated j-invariant; it turns out that the answer is all values in an algebraically closed field.

Proposition 4. *Let $j_0 \in \mathbb{CP}^1$ be an element of the extended complex plane. Then there exists some elliptic curve E over $\mathbb{C}$ for which $j(E) = j_0$.*

Proof. If $j_0 = 1728$, we can pick $y^2 = x^3 + x$, and if $j_0 = \infty$, we can pick $y^2 = x^3$, so assume $j_0 \neq 1728, \infty$. Then we want $A, B \in \mathbb{C}$ such that

$$j_0 = -1728 \frac{(4A)^3}{-16(4A^3 + 27B^2)} = 6912 \frac{A^3}{4A^3 + 27B^2}.$$

If we let $B = 1$, then we are solving

$$\frac{j_0}{6912}(4A^3 + 27) = A^3 \implies \frac{j_0}{1728}A^3 + \frac{j_0}{256} = A^3 \implies \frac{j_0 - 1728}{1728}A^3 + \frac{j_0}{256} = 0,$$

and by the cubic formula there exists a value A that satisfies this equation, and hence there exists an E for which $j(E) = j_0$. □

With these two propositions, we can now see what the moduli space of elliptic curves should look like.

Theorem 5. *The moduli space of isomorphism classes of elliptic curves is $\mathbb{CP}^1$, the one-dimensional complex projective plane. In other words, $\mathcal{M}_1 = \mathbb{CP}^1$.*

Proof. We need show that the set of isomorphism classes of elliptic curves is in one-to-one correspondence with $\mathbb{CP}^1$. But Proposition 3 tells us that these isomorphism classes correspond precisely to distinct values of the j-invariant. It follows by Proposition 4 that the j-invariant can take on all values in $\mathbb{CP}^1$, so the isomorphism classes are bijective with $\mathbb{CP}^1$ as desired. □

4. Acknowledgements

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References

[1] Joseph H. Silverman, *The Arithmetic of Elliptic Curves.* Springer, 2009, 2nd edition.
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[3] Dimitri Zvonkine. *An introduction to moduli spaces of curves and its intersection theory*, available at `https://www-fourier.ujf-grenoble.fr/sites/ifmaquette.ujf-grenoble.fr/files/ete2011-zvonkine.pdf`.