A Survey of Graph Theory

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Irvington Math Club

January 17, 2018

Outline

Graph Theory Teasers

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Consider the map of Konigsberg colorized, circa 1736, presented below. Find a walk through the city that crosses each of the bridges once and only once.

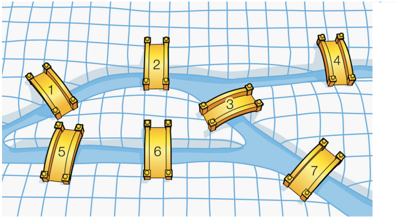

Figure: Bridges of Kongsberg

Alice, Bob, and Carl live in a two-dimensional village and need to be connected with each of the one sources of heat, water, and electricity. Is there a way to make all nine connections without any of the lines overlapping?

Definitions

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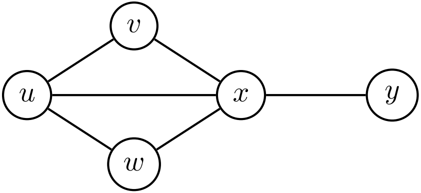

Figure: A graph of order 5 and size 6.

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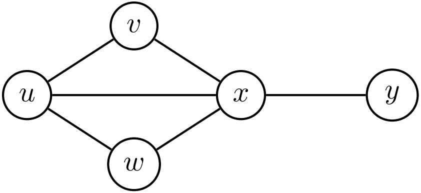

Figure: A graph of order 5 and size 6.

In Figure 6, $V(G) = \{u, v, w, x, y\}$ and $E(G) = \{uv, ux, uw, vx, wx, xy\}$. If $e = uv$ is an edge of G, then u and v are said to be **joined** by the edge e. In this case, u and v are referred to as **neighbors** of each other.

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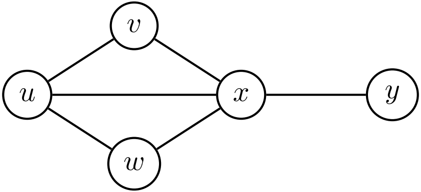

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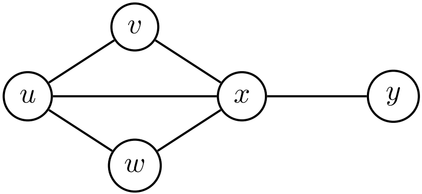

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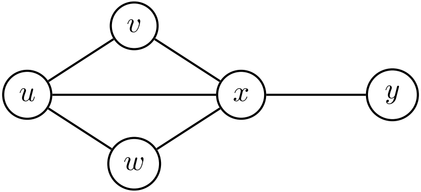

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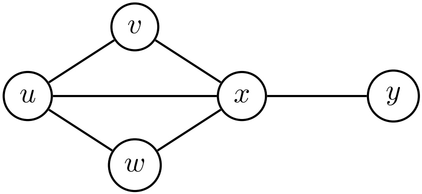

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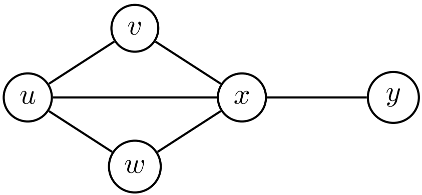

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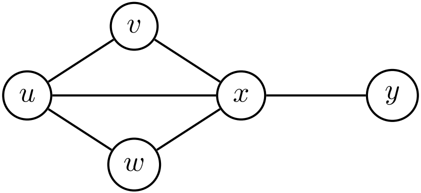

$$\deg u + \deg v + \deg w + \deg x + \deg y$$
$$= 3 + 2 + 2 + 4 + 1 = 2 \cdot 6 = 2m$$

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With this, we can easily characterize graphs possessing an Eulerian trail.

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A connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at another.

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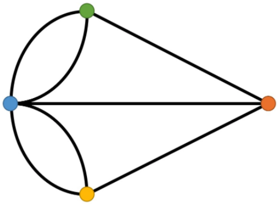

Figure: Konigsberg, modernized

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Theorem (Dirac)

Let G be a graph of order $n \geq 3$. If $\deg v \geq n/2$ for each vertex v of G, then G is Hamiltonian.

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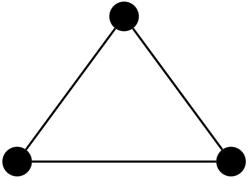
This number is now known as the **Euler Characteristic**, and is referred to with the greek letter χ.

Let's Calculate It!



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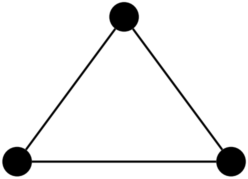
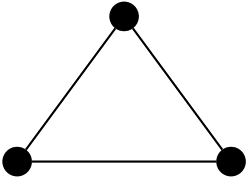


$V = 3$

Let's Calculate It!



$V = 3$

$E = 3$

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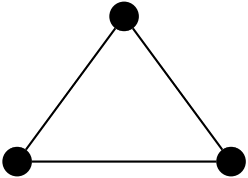
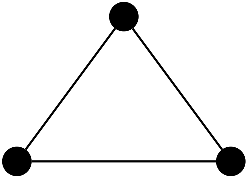


$V = 3$

$E = 3$

$R = 2$

Let's Calculate It!



$$V = 3$$
$$E = 3$$
$$R = 2$$
$$\chi = V - E + R$$

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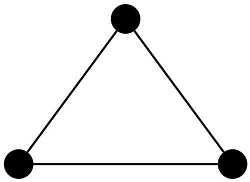

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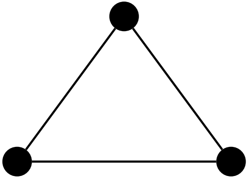

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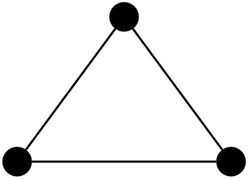

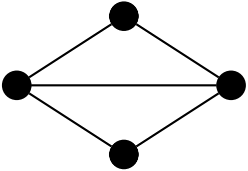

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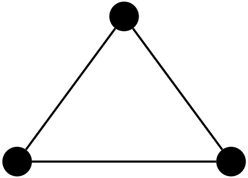

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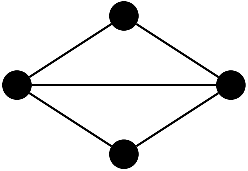

$$V = 4$$

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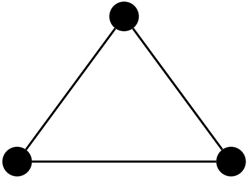

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$$E = 3$$
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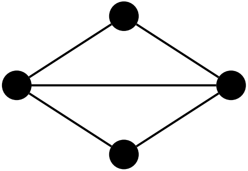
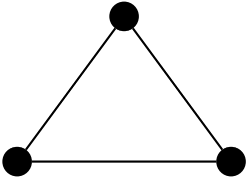
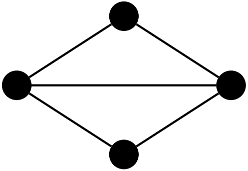


$$V = 4$$
$$E = 5$$

Let's Calculate It!





$$V = 3$$
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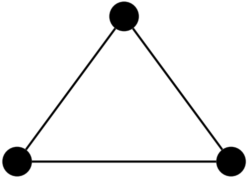

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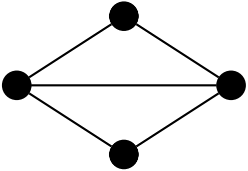

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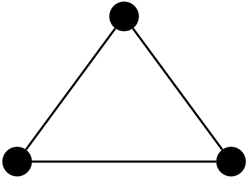

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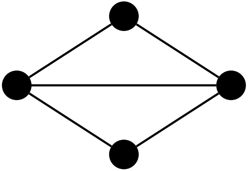

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Come up with your own graphs. Is χ always equal to 2?

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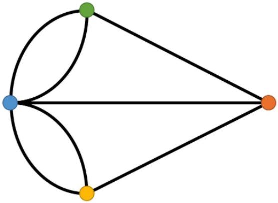

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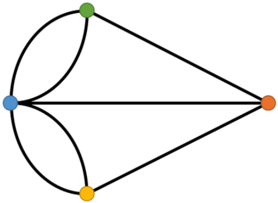

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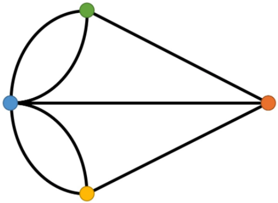

$$V = 4$$
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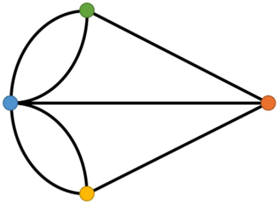

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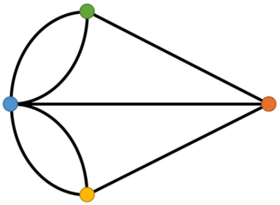

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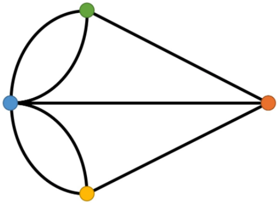

$$V = 4$$
$$E = 7$$
$$R = 5$$
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$$= 4 - 7 + 5 = 2$$

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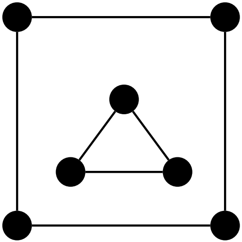

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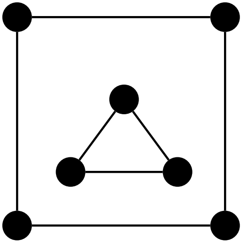

$$V = 7$$

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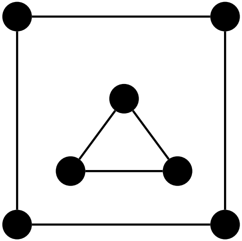

$V = 7$

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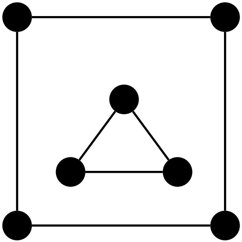
Is it always 2?

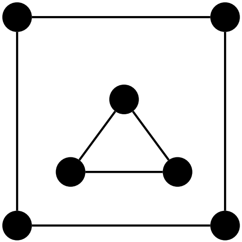


$V = 7$

$E = 7$

$R = 3$

Is it always 2?



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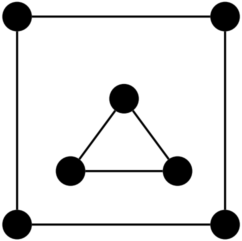

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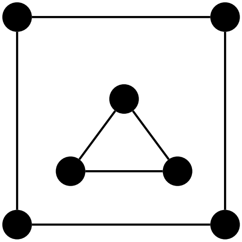

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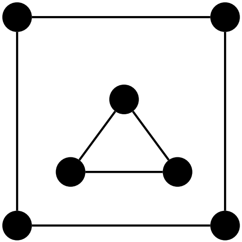
Is it always 2?



$$V = 7$$
$$E = 7$$
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$$\chi = V - E + R$$
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This graph is not **connected**. We can connect the two components by adding an edge.

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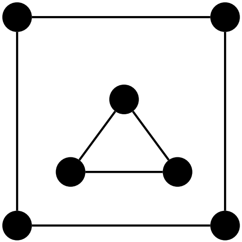

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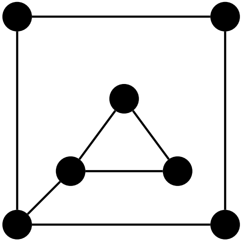

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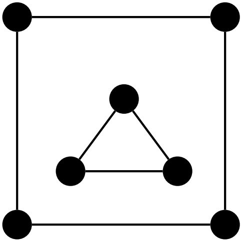

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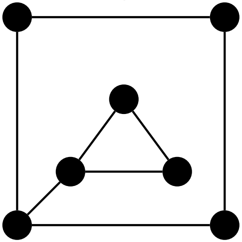

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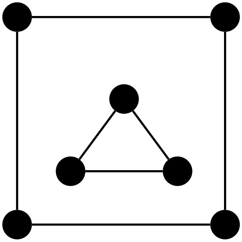

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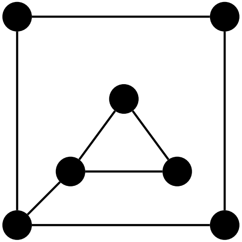
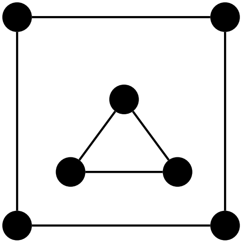


$$V = 7$$
$$E = 8$$

Is it always 2?



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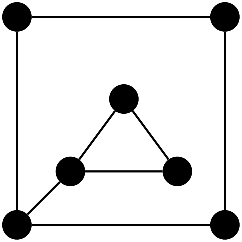
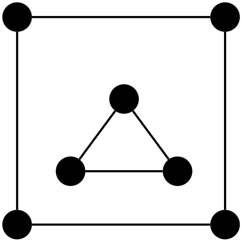


$$V = 7$$
$$E = 8$$
$$R = 3$$

Is it always 2?



$V = 7$

$E = 7$

$R = 3$

$$\chi = V - E + R = 7 - 7 + 3 = 3$$

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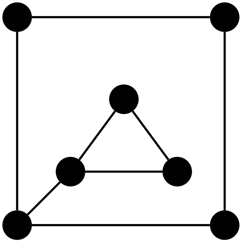

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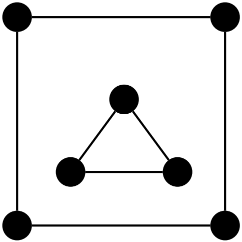

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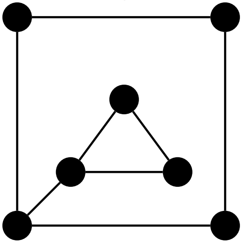

$V = 7$

$E = 8$

$R = 3$

$$\chi = V - E + R$$
$$= 7 - 8 + 3 = 2$$

Ok, are there any other cases?

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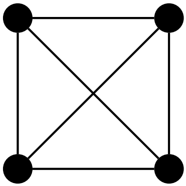

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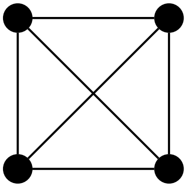
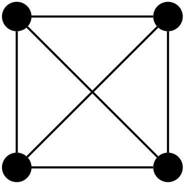


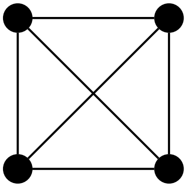
$V = 4$

Ok, are there any other cases?



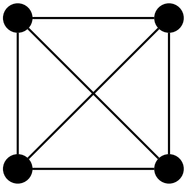
$$V = 4$$
$$E = 6$$

Ok, are there any other cases?



$$V = 4$$
$$E = 6$$
$$R = 5$$

Ok, are there any other cases?



$$V = 4$$
$$E = 6$$
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$$\chi = V - E + R$$

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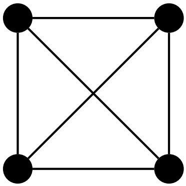

$$V = 4$$
$$E = 6$$
$$R = 5$$
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$$= 4 - 6 + 5 = 3$$

Let's make some modifications

In this graph, the diagonal edges cross. Let's modify it into a **planar** graph, or a graph with no edge crossings.

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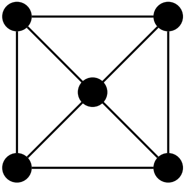
We can add a vertex.



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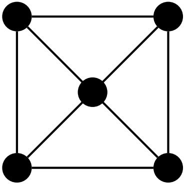

$V = 5$

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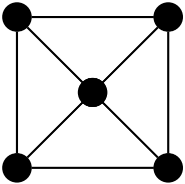

$V = 5$

$E = 8$

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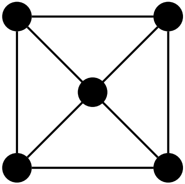

$V = 5$

$E = 8$

$R = 5$

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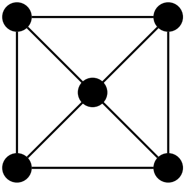

$$V = 5$$
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$$R = 5$$
$$\chi = V - E + R$$

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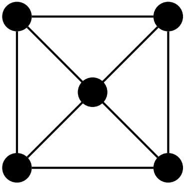

$$V = 5$$
$$E = 8$$
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$$\chi = V - E + R$$
$$= 5 - 8 + 5 = 2$$

Let's make some modifications

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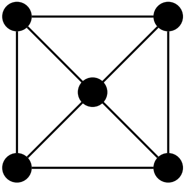

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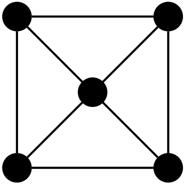

$$V = 5$$
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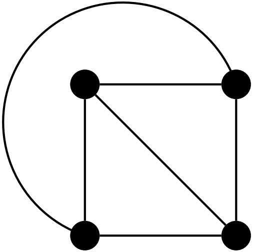
We can move an edge.



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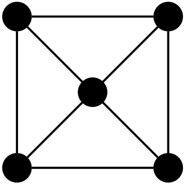

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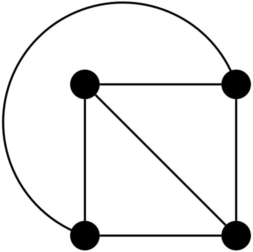

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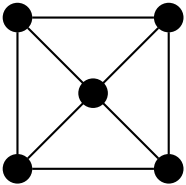

$$V = 5$$
$$E = 8$$
$$R = 5$$
$$\chi = V - E + R$$
$$= 5 - 8 + 5 = 2$$

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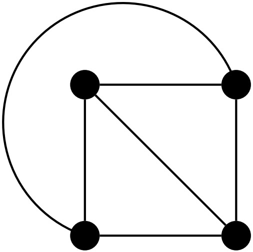

$$V = 4$$
$$E = 6$$

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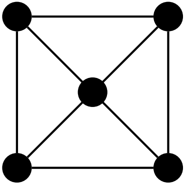

$$V = 5$$
$$E = 8$$
$$R = 5$$
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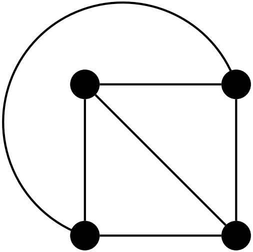
$$= 5 - 8 + 5 = 2$$

We can move an edge.



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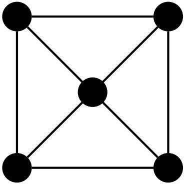

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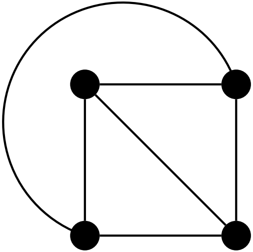

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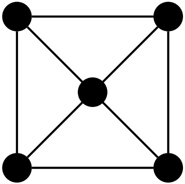
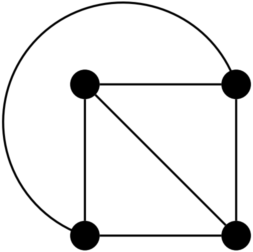


$$V = 5$$
$$E = 8$$
$$R = 5$$
$$\chi = V - E + R = 5 - 8 + 5 = 2$$

We can move an edge.



$$V = 4$$
$$E = 6$$
$$R = 4$$
$$\chi = V - E + R = 4 - 6 + 4 = 2$$

In Conclusion

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Every connected, planar graph has $\chi = V - E + F = 2$.

Three Utilities Problem

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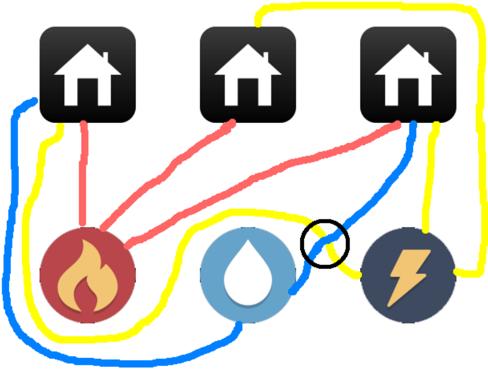

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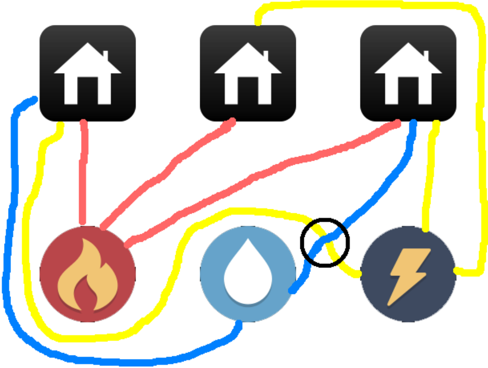

But it turns out this is impossible, since the graph you are asked to draw, $K_{3,3}$, is nonplanar.

Three Dimensional Solids

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We can also consider the Platonic solids

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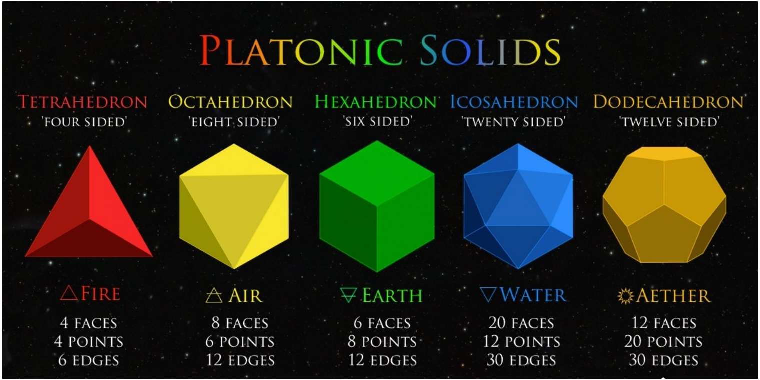
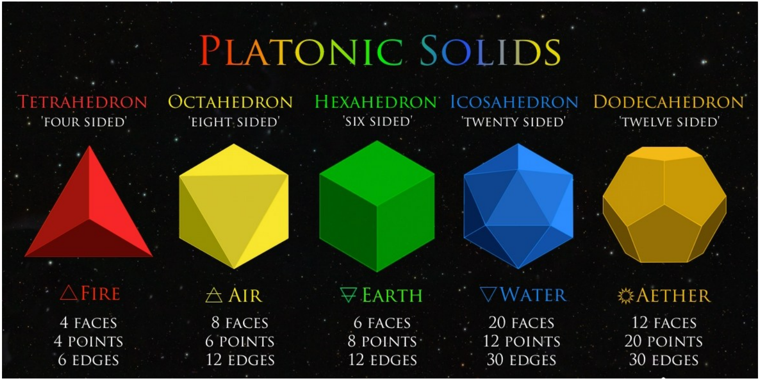


Three Dimensional Solids

We can also consider the Platonic solids



What do you notice about $V - E + F$ for each of these solids?

Surface Invariants

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It turns out that the Euler Characteristic can be used to tell the difference between a sphere and a torus!

Surface Invariants

It turns out that the Euler Characteristic can be used to tell the difference between a sphere and a torus!
In the future, we may see how χ is an example of a **surface invariant**, or a quantity that remains the same for similar shapes (up to homeomorphism), yet is different for distinct shapes (think donut vs. a ball).