Graph Theory

Irvington Math Club

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1 Teasers

Theorem 1 (The Party Theorem)**.** At any party, there is a pair of people who have the same number of friends present.

Problem 2 (Bridges of Konigsberg)**.** Consider the map of Konigsberg colorized, circa 2017, presented below. Find a walk through the city that crosses each of the bridges once and only once.

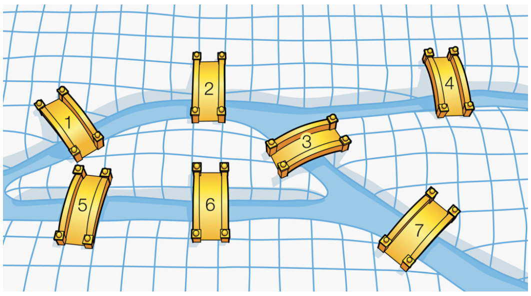

Figure 1: Bridges of Kongsberg

Figure 2: Can you draw all nine lines?

Problem 3 (Three Utilities)**.** Alice, Bob, and Carl live in a two-dimensional village and need to be connected with each of the one sources of water, gas, and electricity. Is there a way to make all nine connections without any of the lines overlapping?

Theorem 4 (Four Color Theorem)**.** At most four colors are needed to color a map so that every region is a different color from its neighbors.

A **graph** G consists of **vertices** or **nodes**, the points, and **edges**, the lines. We frequently write $V(G)$ and $E(G)$ for the vertex and edge sets of G respectively, and call $|V(G)|$ and $|E(G)|$ the **order** and **size** of the graph respectively.

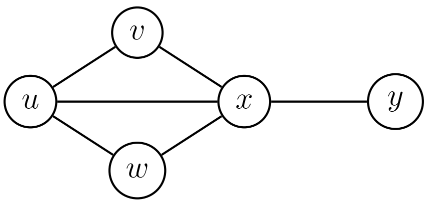

Figure 3: A graph of order 5 and size 6.

In Figure 3, $V(G) = \{u, v, w, x, y\}$ and $E(G) = \{uv, ux, uw, vx, wx, xy\}$. If $e = uv$ is an edge of G, then u and v are said to be **joined** by the edge e. In this case, u and v are referred to as **neighbors** of each other.

2 Degree Sequences and the Havel-Hakimi Theorem

The **degree** of a vertex v, $\deg v$, is the number of neighbors of v.

Theorem 5 (Party Theorem, rephrased)**.** In a graph of order n, at least two vertices have the same degree.

Theorem 6 (First Theorem of Graph Theory)**.** If G is a graph of size m, then the sum of the degrees of every vertex is equal to $2m$.

If the degrees of the vertices of a graph G are listed in a sequence s, then s is called a **degree sequence** of G. For example, all of the sequences

$$s : 3, 2, 2, 3, 1; \quad s' : 1, 2, 2, 3, 3; \quad s'' : 3, 3, 2, 2, 1$$

are degree sequences of the graph G in Figure 3. Finding degree sequences of a graph is easy. The converse is interesting however. Given a sequence of nonnegative integers, is it a degree sequence of some graph? Such a sequence is called **graphical**.

Problem 7. Which of the following sequences are graphical?

- $s_1 : 3, 3, 2, 2, 1, 1$
- $s_2 : 6, 5, 5, 4, 3, 3, 3, 2, 2$
- $s_3 : 7, 6, 4, 4, 3, 3, 3$

The following theorem makes it easy to determine if a sequence is graphical.

Theorem 8 (Havel-Hakimi)**.** A non-increasing sequence $s : d_1, d_2, \ldots, d_n (n \geq 2)$ of non-negative integers, where $d_1 \geq 1$, is graphical if and only if the sequence

$$s_1 : d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$$

is graphical.

3 Eulerian and Hamiltonian Graphs

For a connected graph G, any open trail that contains every edge of G is an **Eulerian trail**. If G contains a closed Eulerian trail, it is **Eulerian**

Theorem 9. A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.

With this, we can easily characterize graphs possessing an Eulerian trail.

Corollary 10. A connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at another.

Problem 11. Prove no such walk exists crossing each of the bridges of Konigsberg once.

There is an analog to Eulerian trails. A path in a graph G that contains every vertex of G is called a **Hamiltonian path**.[1] Unfortunately, these are much less well-behaved. Here is a simple sufficient condition for a graph to be Hamiltonian, proven in 1952.

Theorem 12 (Dirac)**.** A graph with n vertices ($n \geq 3$) is Hamiltonian if every vertex has degree $\geq n/2$.

[1] We studied Eulerian *trails* because such trails may need to have repeated vertices, unlike paths which necessarily have unique vertices.

4 Planar Graphs and Kuratowski's Theorem

A graph G is called a **planar graph** if G can be drawn in the plane so that no two of its edges cross each other. If G is planar, then it divides the plane into pieces called **regions**. We have the following famous result:

Theorem 13 (The Euler Identity)**.** If G is a planar graph of order n, size m with r regions, then $n - m + r = 2$.

You can imagine how hard it is to prove a graph to be non-planar: you can't possibly check every way of drawing the graph. Euler's Identity leads to some simple conditions on planarity.

Theorem 14. If G is a planar graph of order $n \geq 3$ and size m, then $m \leq 3n - 6$.

Problem 15. Show that the graph of order 5 with edges between every pair of vertices (called the complete graph on 5 nodes, or K_5) is nonplanar.

It turns out that setup in Problem 3 is indeed impossible, since the graph is nonplanar. The graph with six vertices, where three vertices are connected to all three other vertices, is the complete bipartite graph $K_{3,3}$.

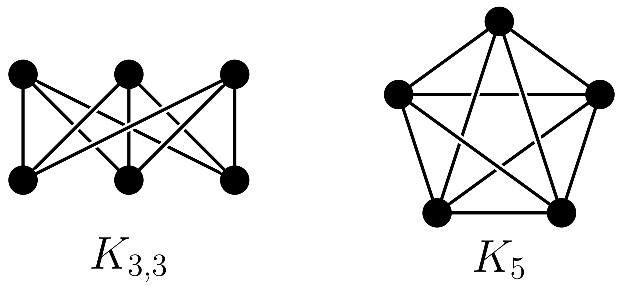

Figure 4: The Kuratowski graphs

Theorem 16 (Impossibility of the Three Utilities Problem)**.** The graph $K_{3,3}$ is nonplanar.

We can already see how much trouble it can be to determine planarity for arbitrary graphs, let alone simple ones like $K_{3,3}$ and K_5. Surprisingly, the main enemies to planarity are precisely these two graphs, which leads to a simple check to see if a graph is planar, discovered in 1930. Before I can state Kuratowski's remarkable theorem, I need to state two notions.

A **subdivision** of a graph is constructed by replacing one edge by two edges with a vertex in between, and a **subgraph** is constructed by removing vertices or edges.

Theorem 17 (Kuratowski)**.** A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

In other words, G is planar if it is not possible to subdivide the edges of K_5 or $K_{3,3}$, and then possibly add edges or vertices, to get G.