Counting

Irvington Math Club

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There's two problems that you should tackle during lecture; pick one (or both) to solve, and we'll go over one of them. The rest are for homework. Email me at `tyler.zhu@berkeley.edu` if you want explanations to the other ones.

1 Stars and Bars (Teaser)

Problem 1

How many cubic polynomials $f(x)$ with positive integer coefficients are there such that $f(1) = 9$?

Solution. Let $f(x) = ax^3+bx^2+cx+d$. The only condition on $f(x)$ is that $f(1) = 9$, which means $a + b + c + d = 9$. Since a, b, c, d are positive integers, there are $\binom{8}{3} = 56$ such polynomials. □

Problem 2 (HMMT 2017)

How many ways are there to insert +'s between the digits of 111111111111111 (fifteen 1's) so that the result will be a multiple of 30?

Solution. No matter how many +'s we insert, the result will always be a multiple of 3 since there are fifteen 1's. For it to be a multiple of 10, we need exactly 10 numbers, which means we're adding 9 +'s. There are 14 gaps, so our answer is $\binom{14}{9}$. □

Exercise 1.1. How many solutions are there in positive integers to the equation $w+x+y+z = 30$ if no variable takes on a value greater than 16?

Exercise 1.2 (Hockeystick)**.** Prove that

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

with a combinatorial argument. Hint: count the positive integer solutions to $x_1 + \cdots + x_{r+1} \leq n + 1$ in two ways.

Exercise 1.3 (10A 2016)**.** When $(a + b + c + d + 1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables $a, b, c,$ and d, each to some positive power. What is N?

2 Double Counting and 1-1 Correspondences

Problem 3 (Moscow 1963)

Let $a_1, a_2, \ldots, a_n$ be a sequence of arbitrary natural numbers. Define b_k to be the number of elements a_i for which $a_i \geq k$. Prove that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots$.

Solution. The idea is to double count. Drawing a picture is the best way to see this: for each a_i, draw a_i circles vertically. Then the LHS is counting the number of circles going vertically, while the RHS is counting them horizontally.

Formally, one way to count is simply to sum the a_i's. Another way to count uses the fact that they are natural numbers. All the b_i start at 0. Then, for any given a_i, b_1 through b_{a_i} will have all their values increased by 1, increasing the total on the RHS by a_i. Hence the total contributions of all a_i to the RHS is just $a_1 + \cdots + a_n$. $\square$

Problem 4 (HMMT 2006)

Compute

$$\sum_{n_{60}=0}^{2} \sum_{n_{59}=0}^{n_{60}} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} \sum_{n_0=0}^{n_1} 1.$$

Solution. Another way of phrasing the problem is to find the number of solutions to $0 \leq n_0 \leq n_1 \leq n_2 \leq \cdots \leq n_{60} \leq 2$. This corresponds to the number of right-up walks on a 61×2 grid from the bottom left to the top right, which we all know to be $\binom{63}{2}$.

Alternatively, notice that every solution is of the form $(0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2)$, so we only need to specify the number of 0s, 1s, and 2s. This is equivalent to solving the equation $x+y+z = 61$ for nonnegative x, y, z, which (by stars and bars) has $\binom{63}{2}$ solutions. $\square$

Exercise 2.1 (HMMT 2006)**.** How many ordered triples (a, b, c) of positive integers less than 10 have abc divisible by 20?

Exercise 2.2 (10A #25 2019)**.** For how many integers n between 1 and 50 inclusive is

$$\frac{(n^2-1)!}{(n!)^n}$$

an integer? Hint: Show that $\frac{(n^2)!}{(n!)^{n+1}}$ is always an integer by counting.

Exercise 2.3. How many ways can we color each square of a 2007×2007 square grid either black or white such that each row and each column has an even number of black squares?

3 Recurrences

This is the only section I'll write stuff for since some of you may be new to it.

The idea is that we'll have a quantity we care about (say length 10 strings with no repeated letters) that can actually be built up from smaller instances of the same thing (length 8 and 9 strings with no repeated letters). So we can create a formula and then just plug and chug away.

They're called recurrences because these formulas often depend on themselves. Fibonacci is a good example of this: $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 1, F_1 = 1$. In fact Fibonacci numbers show up very often in these problems.

Example 3.1

How many bit strings of length 10 have at least 3 consecutive 1's?

Solution. We'll count complements instead, looking at how many bit strings of length 10 have no more than two consecutive 1's.

Let $b(n) = \#$ of bit strings of length n with no more than two consecutive 1's. If we try to construct a recurrence for this, we see that either we add a 0 at the end of any $n-1$ length string, a 01 at the end of a $n-2$ length string, or a 011 at the end of a $n-3$ length string. Hence, this means that $b(n) = b(n-1) + b(n-2) + b(n-3)$. We know that $b(1) = 2$, $b(2) = 4$, and $b(3) = 7$, so working our way to $b(10)$ gives us 504. Subtracting from $2^{10} = 1024$ gives 520. □

Another toy example to practice on: show that the number of bit strings of length n with no consecutive 1's is F_{n+1}.

In the following problems, you may need to make multiple recurrences that depend on each other. Good luck!

Problem 5 (AIME II 2015)

There are 2^{10} possible 10-letter strings in which each letter is either an A or a B. How many such strings do not have more than 3 adjacent identical letters?

Solution. Let a_n be the number of n-letter strings starting with A subject to the above constraints, and b_n be the same but with strings starting with B. Then to get a length n string starting with A, we can add A to a $n-1$ length string starting with B, add AA to a $n-2$ length string starting with B, or add AAA to a $n-3$ length string starting with B. So $a_n = b_{n-1} + b_{n-2} + b_{n-3}$.

In the same way, we find $b_n = a_{n-1} + a_{n-2} + a_{n-3}$. Now since we're interested in the total number of strings, we're really looking for $a_n + b_n$. So let $s_n = a_n + b_n$. We get $s_n = s_{n-1} + s_{n-2} + s_{n-3}$, where $s_1 = 2, s_2 = 4, s_3 = 8$. Working our way up gives us $s_{10} = 548$. □

Problem 6 (HMMT Guts 2011)

In how many ways can each square of a 4×2011 grid be colored red, blue, or yellow such that no two squares that are diagonally adjacent are the same color?

Official Solution. If we first color the board in a checkerboard pattern, it is clear that the white squares are independent of the black squares in diagonal coloring, so we calculate the number of ways to color the white squares of a $4 \times n$ board and then square it.

Let a_n be the number of ways to color the white squares of a $4 \times n$ board in this manner such that the two squares in the last column are the same color, and b_n the number of ways to color it such that they are different. We want to find their sum x_n. We have $a_1 = 3, b_1 = 6$.

Given any filled $4 \times n - 1$ grid with the two white squares in the last column different, there is only 1 choice for the middle square in the nth row, and two choices for the outside square, 1 choice makes them the same color, 1 makes them different. If the two white squares are the same, there are 2 choices for the middle square and the outer square, so 4 choices. Of these, in 2 choices, the two new squares are the same color, and in the other 2, the two squares are different. It follows that $a_n = 2a_{n-1} + b_{n-1}$ and $b_n = 2a_{n-1} + b_{n-1}$, so $a_n = b_n$ for $n \geq 2$. We have $x_n = 8 \cdot 3^{n-1}$ and $x_{2011} = 8 \cdot 3^{2010}$. So the answer is $64 \cdot 3^{4020}$. □

Exercise 3.2 (10B #25 2019)**.** How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

Exercise 3.3 (HMMT 2014)**.** We have a calculator with two buttons that displays an integer x. Pressing the first button replaces x by $\lfloor \frac{x}{2} \rfloor$, and pressing the second button replaces x by $4x + 1$. Initially, the calculator displays 0. How many integers less than or equal to 2014 can be achieved through a sequence of arbitrary button presses? (It is permitted for the number displayed to exceed 2014 during the sequence. Here, $\lfloor y \rfloor$ denotes the greatest integer less than or equal to the real number y.)

Exercise 3.4 (NIMO)**.** Let $S = \{1, 2, \cdots, 2013\}$. Let N denote the number of 9-tuples of sets $(S_1, S_2, \dots, S_9)$ such that $S_{2n-1}, S_{2n+1} \subseteq S_{2n} \subseteq S$ for $n = 1, 2, 3, 4$. Find the remainder when N is divided by 1000.

4 Be Smart

Problem 7 (12A #18 2010)

A 16-step path is to go from $(-4, -4)$ to $(4, 4)$ with each step increasing either the x-coordinate or the y-coordinate by 1. How many such paths stay outside or on the boundary of the square $-2 \le x \le 2$, $-2 \le y \le 2$ at each step?

Solution. By symmetry, we will assume the path goes right, then up, and multiply by two because we can reflect the path along the line $y = x$.

There are three "safe" points that a path must go through to be considered valid: $(2, -2)$, $(3, -3)$, and $(4, -4)$. Note that a path can only go through one of the aforementioned points, but must go through one such point. The number of ways to go through these points is $\binom{8}{2}^2 + \binom{8}{1}^2 + \binom{8}{0}^2 = 784 + 64 + 1 = 849$. Plus symmetry, the answer is $849 \times 2 = 1698$. □

Problem 8 (CHMMC 2010)

Cindy draws randomly from a box with 2010 red balls and 1957 blue balls, one ball at a time without replacement. She wins if, at anytime, the total number of blue balls drawn is more than the total number of red balls drawn. Assuming Cindy keeps drawing balls until she either wins or runs out, compute the probability that she eventually wins.

Solution. Suppose we draw all of the balls. Label the red balls $R_1, R_2, \dots, R_{2010}$ and the blue balls $B_1, B_2, \dots, B_{1957}$ (left to right). Note that if B_{1957} is located to the right of R_{1957} then the arrangement is not winnable. So, B_{1957} must be located to the left of R_{1957}. It is clear that this arrangement is winnable regardless of how the other balls are arranged because $1957 > 1956$. Since there are 2011 slots this occurs with probability $\frac{1957}{2011}$. □

Exercise 4.1 (AIME 2006)**.** Let $(a_1, a_2, \dots, a_{12})$ be a permutation of $(1, 2, \dots, 12)$ for which

$$a_1 > a_2 > a_3 > a_4 > a_5 > a_6 \text{ and } a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}.$$

An example of such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$. Find the number of such permutations.

Exercise 4.2. Let N denote the number of 7-tuples of sets $S_1, S_2, \dots, S_7$, not necessarily distinct, for which

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_7 \subseteq \{1, 2, 3, 4, 5, 6, 7\}.$$

Find the remainder when N is divided by 1000.

Exercise 4.3 (AIME 1990)**.** In a shooting match a marksman must break eight targets arranged in three hanging columns of 3, 3 and 2 targets respectively. Whenever a target is broken, it must be the lowest unbroken target in its column. In how many different orders can the eight targets be broken?