Combinatorics Redux

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March 27, 2019

The main purpose of this handout is to demonstrate the many uses of the balls and boxes technique. My hope is that organizing the topics in this fashion will make it easier to see how they are all related. Poorly named title I know, but at least it sounds cool.

1 Balls and Boxes

1.1 The Technique

The basic setup of balls and boxes is already covered in the notes, so I won't try to do better than it. But I can offer another way to think about it using stars and bars.

Example 1.1. How many ways can I distribute 3 balls among 3 unique boxes?

Let's explore this problem a little bit. If I try writing all of the possibilities out, eventually I'll get something like this.

<table>
<tr><th>#</th><th>Distribution</th><th>Stars and Bars</th></tr>
<tr><td>1</td><td>[ ] [ ] [○○○]</td><td>||***</td></tr>
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The answer of course is 10. But instead of writing out every single distribution, we can be more clever and use stars and bars. The idea is to write the distribution in terms of the things to be divided (stars) and the dividers (bars). Here, we have three balls to be divided which will be our stars, and we want to distribute them among 3 boxes. Separating into three chunks requires two dividers, so we have 3 stars and 2 bars.

All that's left now is to count. Notice that for every rearrangement of these 3 stars and 2 bars, we get a distinct distribution corresponding to it (look at the table). But we know how to count rearrangements. There are 5! total ways to rearrange them, but we divide by 2! for permutations of the bars and 3! for permutations of the stars, for a total of $\frac{5!}{2!3!} = \binom{5}{2} = \boxed{10}$.

Here's another way of counting the same thing. We have n balls to be distributed among k unique boxes. This means there are n stars needing $k-1$ bars, so we count the number of strings of n stars and $k-1$ bars. Imagine having $n+k-1$ empty spots. We can choose $k-1$

spots to be bars and let the rest be stars, which amounts to $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$[1] such strings, and hence distributions.

While balls and boxes is named after this situation, it can occur in many different fashions; here are just a few of them.

Problem 1. What is the number of nonnegative solutions to $x_1 + x_2 + \cdots + x_5 = 10$?

Problem 2. If I have ten pencils, how many ways can I hand them out to five kids?

Problem 3. How many numbers in between 0 and 99999 have digits that sum to 10?

In fact, all of these problems have the same answer, $\binom{14}{4}$. This shows that how you interpret the problem and set up your method of counting can often be the only hard part of the problem.

1.2 Variations

The three problems I just showed all have a common theme: we have things to distribute into categories, and each category could have no items given to it. That's the most typical situation. Now let's look at some common variations on it.

Problem 4. Steph has 10 balls to give out to 5 kids, but he doesn't want to make any of them sad so he wants every kid to get at least one. How many ways can he hand them out?

Here we have a nonzero restriction. I'll present one way of tackling it here, and present the other one in Section 2.

Solution. As usual, we have 10 stars and $5 - 1 = 4$ bars. This time, imagine having a gap between every two stars as pictured:

$$* \_ * \_ * \_ * \_ * \_ * \_ * \_ * \_ * \_ *$$

If we pick a different gap to put each bar in, then we can guarantee each chunk will have a star in it. For example, one way to put bars in is

$$* \_ * \mid * \_ * \_ * \mid * \mid * \_ * \mid * \_ *$$

which would correspond to giving the kids $2, 3, 1, 2, 2$ balls respectively. We have $10 - 1 = 9$ gaps, and we need to choose 4 to put the bars in, which gives $\binom{9}{4} = \boxed{126}$. □

There's also another common variation, which is having transformed variables like even or odd amounts, or "greater than 1" conditions. These can usually be taken care of with a simple change of variables.

Problem 5 (Fa16 MT2)**.** How many combinations of even natural numbers (including zero) (x_1, x_2, x_3, x_4) are there such that $x_1 + x_2 + x_3 + x_4 = 20$?

Solution. We know that even numbers are of the form $2k$, so if we divide through our equation by 2, we'll get $x_1' + x_2' + x_3' + x_4' = 10$ where the (x_1', x_2', x_3', x_4') are just plain natural numbers without restrictions. For every solution of this form, we can get a solution in our desired form of (x_1, x_2, x_3, x_4) by multiplying by 2 (in other words, the two types of solutions are in bijection). Hence, using balls and boxes on our new equation gives $\binom{10+4-1}{4-1} = \binom{13}{3}$. □

There's a final variation which isn't as common, but it's tricky and elegant enough that I felt like should be included. It has to do with multiple instances of balls and boxes.

Problem 6. How many ways can I hand out up to 8 identical midterms to 5 students?

[1] We used $\binom{n}{k} = \binom{n}{n-k}$ here. Why is this true? Prove it combinatorially.

Solution. Suppose I choose to hand out k midterms, where $0 \le k \le 8$. Then this is just standard stars and bars, where I have k stars to be split by 4 bars for a total of $\binom{k+4}{4}$. Now I just sum this over k and get $\sum_{k=0}^{8} \binom{k+4}{4}$ which is... um... yeah how do you evaluate this?

Let's think of a smarter way to do this. Notice that no matter how many midterms I hand out, I'm left with the remaining $8 - k$ midterms. This means we can think of it as another distribution problem, but distributing the 8 midterms to 6 people; the 5 students and myself. Solving this reformulation, we have 8 stars to be split by 5 bars for a total of $\binom{8+5}{5} = \binom{13}{5}$. □

I should mention that you can use the previous technique to prove the well-known Hockeystick Identity; this is one of the exercises.

2 Dealing with Constraints

The final (kind of annoying) variation of balls and boxes is when the boxes have a restriction on the number of balls. The idea here is to figure out a smart way of counting the constraints, either by dealing some balls out in advance to certain boxes and/or by counting the bad cases and subtracting them.

I'll first illustrate the technique on a problem we've already seen.

Example 2.1. Steph has 10 balls to give out to 5 kids, but he doesn't want to make any of them sad so he wants every kid to get at least one. How many ways can he hand them out?

Solution. To make the kids happy, he first gives out a ball to each kid, so that they are all already satisfied. This reduces the problem to distributing $10 - 5 = 5$ balls to five kids without any constraints, which is our classic balls and boxes. This gives $\binom{5+5-1}{5-1} = \binom{9}{4} = \boxed{126}$. □

Seem familiar? We did a similar thing before to deal with awkward restrictions on the number of balls. However, we will now use this to account for overlapping events. Here's a simple example.

Example 2.2. How many ways can we distribute 8 balls among 4 distinguishable boxes if the first box can contain at most 5 balls?

Solution. If there was no restriction, then the total would be $\binom{8+4-1}{4-1} = \binom{11}{3}$. But this includes the ases where the first box contains 6 or more balls, which are illegal. So let's do the same trick and set aside 6 balls for the first box. Now we're distributing 2 balls among 4 boxes, which gives $\binom{2+4-1}{4-1} = \binom{5}{3}$. Subtracting these bad cases from our total leaves us with $\binom{11}{3} - \binom{5}{3}$. □

Here's another problem with the same idea, but a little trickier.

Problem 7 (Sp16 MT2)**.** How many ways are there to split up n dollars among r friends where each friend gets at least 1 dollar and no friend gets more than half of the dollars. (You may assume that n is even and $r \ge 3$.)

Solution. First, if there was no restriction, then our total would just be $\binom{n-1}{r-1}$. What are the bad cases here? When any person has more than $\frac{n}{2}$ dollars. So if we set aside $\frac{n}{2}$ dollars to someone, then after distributing, they will be guaranteed to have over $\frac{n}{2}$ dollars. In fact, only that person has a chance of getting more than half, since we won't have enough dollars to give someone else more than half afterwards (this is why $r \ge 3$ is helpful).

The number of ways to distribute this way is $\binom{\frac{n}{2}-1}{r-1}$, but there's r choices for the person to give $\frac{n}{2}$ dollars to. Overall, our answer then is $\binom{n-1}{r-1} - r\binom{\frac{n}{2}-1}{r-1}$. □

3 Summary

In summary, when you do balls and boxes problems, be aware of the following details:

- Are the balls unique or not? If so, we don't need to use balls and boxes.
- Is each box getting at least one ball, or can some of them have none?
- Do the variables have some condition like even or greater than 0 that we can get rid of easily? (i.e. by dividing by 2 or subtracting by 1)
- Are there restrictions on the number of balls in each box? If so, we will need to count the bad cases, and maybe appeal to the Principle of Inclusion-Exclusion (see Ex. 4.8).

In particular, the last case is always dealt with by looking at the restrictions and satisfying them to reduce the problem to another simple case of balls and boxes.

4 Exercises

Exercise 4.1. How many ways can I distribute k unique balls among n unique boxes?

Exercise 4.2 (Fa15 MT2)**.** How many ways are there to split up 10 dollars among Bob, Alice and Eve?

Exercise 4.3 (CSM18)**.** How many ways are there to arrange the letters of the word SUPERMAN such that SUPER occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other) ...

(a) on a straight line?

(b) on a circle?

Exercise 4.4 (Classic)**.** Answer in terms of N and n.

(a) How many nonnegative integer solutions are there to $x_1 + x_2 + \cdots + x_n = N$?

(b) What if the x_i have to be positive?

(c) What if the $=$ sign is changed to a $\leq$?

(d) What if the $=$ sign is changed to a $\leq$ and the x_i are positive?

Exercise 4.5 (Hockeystick)**.** Prove that $\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$ with a combinatorial argument.

Exercise 4.6 (Sp18 MT2)**.** For a prime p, and $d < p$, how many polynomials in $GF(p)$ (modulo arithmetic modulo p) of degree d are there with exactly d roots? (Here, we assume $(x-2)^2$ has two roots at $x = 2$.)

Exercise 4.7 (Fa14 Final)**.** How many numbers are there from 0 to 999999 whose digits sum to 9? Sum to 19?

Exercise 4.8 (Fa18 MT2 Review)**.** If a bin has at least 5 balls in a bin, the 5 balls will fall out and not be counted (e.g., 6 balls in a bin is the same as 1). Compute the number of ways to distribute 7 indistinguishable balls among 4 bins.

Exercise 4.9 (Hard)**.** I have 5 magical hats that can each fit up to 4 equally magical rabbits. How many ways can I distribute 8 magical rabbits among my hats? Some of the hats could be empty.

Exercise 4.10 (AMC)**.** When $(a + b + c + d + 1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables $a, b, c,$ and d, each to some positive power. What is N?

5 Hints

Here are some hints to the exercises, along with the answers to some of them.

4.1 The balls are unique here. Answer is n^k.

4.2 Basic balls and boxes will suffice.

4.3 For part (a), think of the letters S,U,P,E,R as the bars and M,A,N as the stars. For part(b), rotations are symmetric, so fix one letter at the top and do the same as in part (a). Answers are $3!\binom{8}{3}$ and $3!\binom{7}{3}$.

4.4 Section 1.2 is all about this. Be careful with enforcing the positive condition in (d) however. Answers are $\binom{N+n-1}{n-1}$, $\binom{N-1}{n-1}$, $\binom{N+n}{n}$, $\binom{N}{n}$.

4.5 Count the number of solutions to $x_1 + \cdots + x_{r+1} \leq n + 1$ for positive $(x_1, \ldots, x_r)$ in two ways.

4.6 Balls and boxes on the choices. We have p boxes, one for every possible root, and d total choices.

4.7 Use stars and bars; the bars separate the digits into each decimal place. For the second part, deal with the constraint by giving 10 to each digit to count the bad cases.

4.8 Deal with the constraint by setting aside 5 balls. Get rid of all the bad cases where balls disappear, but remember to count the disappearing cases; they're all the same!

4.9 Use Principle of Inclusion-Exclusion. Choose a hat to set aside 4 rabbits for, and count the number of bad cases here. But this overcounts the cases where two hats both have 4 rabbits, so add that back.

4.10 We have 4 boxes (a, b, c, d) that need a positive amount of balls. Think of Problem 6