Topics:

- Bilinearity of covariance ✓
- [Fa15] 1(b): Let X be uniform on [0,1]. Find E[X^5]
- [Fa15] 1(c): Let X,Y,Z be iid. Then E[X+Y|X+Y+Z] = (2/3)(X+Y+Z) ✓
- Chebyshev Proof: |X-\mu| > c --> |X-\mu|^2 > c^2 ✓

Chebyshev: r.v. X, $c>0$

$$\mathbb{P}\left(|X-\mathbb{E}[X]|>c\right) \le \frac{\mathrm{Var}(X)}{c^2}.$$

Pf: $|x|>c \iff |x|^2>c^2$. Then use Markov's on $X' = |X-E[X]|^2$.

$\mathbb{E}[X'] = \mathrm{Var}(X)$, so

$$\mathbb{P}(X'>c^2) \le \frac{\mathbb{E}[X']}{c^2} = \frac{\mathrm{Var}(X)}{c^2}. \quad \square.$$

Fa15 1b) Note 16: Functions of R.V.

"LOTUS": X r.v., a function f

$$\mathbb{E}[f(X)] = \sum_{x\in X} \mathbb{P}(X=x)\, f(x).$$

cts X (pdf):

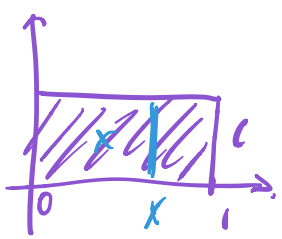
$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} f_X(x)\, g(x)\, dx.$$

$$\mathbb{E}[X] = \sum_{x\in X} \mathbb{P}(X=x)\cdot x$$

$$\mathbb{E}[X^2] = \sum_{x\in X} \mathbb{P}(X=x)\, x^2.$$

$X \sim U[0,1]$

$$f_X(x) = \begin{cases} 0 & x\notin[0,1] \\ 1 & x\in[0,1] \end{cases}.$$



$$\mathbb{E}[X^5] = \int_0^1 f_X(x)\cdot x^5\, dx$$

$$= \int_0^1 x^5\, dx = \frac{x^6}{6}\Big|_0^1 = \frac{1}{6}.$$

$Var(X+Y) \neq Var[X] + Var(Y)$

$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$.

Bilinear: for $a, b \in \mathbb{R}$, then

$$Cov(a \cdot X, b \cdot Y) = ab\, Cov(X,Y).$$

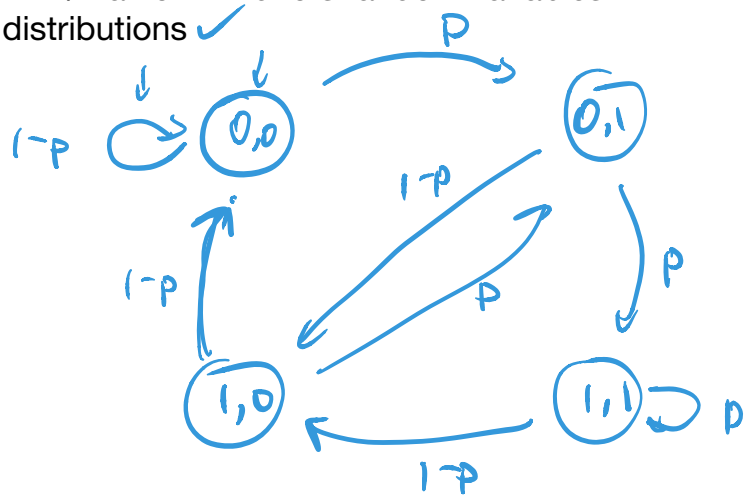
$\vec{x}, \vec{y} \in \mathbb{R}^n$, $(a\cdot\vec{x}) \cdot (b\cdot\vec{y}) = (ab)(\vec{x}\cdot\vec{y})$.

$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

$Cov(aX,Y) = \mathbb{E}[aXY] - \mathbb{E}[aX]\mathbb{E}[Y] = a\,\mathbb{E}[XY] - a\,\mathbb{E}[X]\mathbb{E}[Y]$

$= a\, Cov(X,Y)$. □

- finding the min/max of n i.i.d. cts random variables
- stationary distributions ✓



$\pi = \pi P$ ← balance equations: "how could I have gotten here?"

$\pi(0,0) = (1-p)\pi(0,0) + (1-p)\pi(1,0)$
$\pi(1,0)$;
$\pi(0,1)$;
$\pi(1,1)$.

$\sum \pi(i) = 1$.

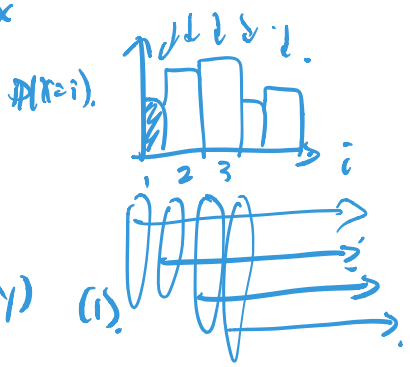
$\pi(0,0) + \pi(0,1) + \pi(1,0) + \pi(1,1) = 1$.

$X_1, \ldots, X_n$, $X_i \sim$ some distribution. Find $\min(X_1, \ldots, X_n) = Y$.

Hwk Q5: $X_i \sim U[0,1]$.

Pf: Tail Sum: $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\,dx$

$\mathbb{E}[X] = \sum_{i=0}^{\infty} \mathbb{P}(X > i)$



$\mathbb{P}(Y > y) \overset{(1)}{=} \mathbb{P}(X_1 > y)\,\mathbb{P}(X_2 > y) \cdots \mathbb{P}(X_n > y)$

$\Rightarrow \overset{(2)}{=} \mathbb{P}(X_1 > y)^n = (1 - F_X(y))^n$. $F_X(y) = \mathbb{P}(X \le y)$.

$\Rightarrow \mathbb{E}[Y] = \int_0^\infty (1 - F_X(y))^n\,dx$.

Uniform: $F_X(x) = x$.

Max. $Z = \max(X_1, \ldots, X_n)$

$\mathbb{P}(Z \le z) = \mathbb{P}(X_1 \le z)\,\mathbb{P}(X_2 \le z) \cdots \mathbb{P}(X_n \le z)$

$= \mathbb{P}(X_1 \le z)^n = F_X(z)^n$.

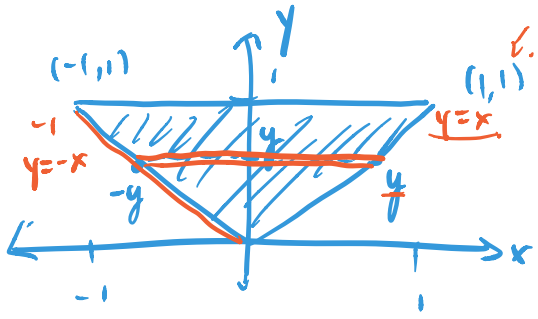
$\frac{dF_Z(z)}{dz} = f_Z(z) = n \cdot F_X(z)^{n-1} \cdot f_X(z)$.

$\mathbb{E}[Z] = \int_{\mathbb{R}} z \cdot f_Z(z)\,dz$

Q: $X_i \sim \text{Expo}(\lambda_i)$, $X_1, \ldots, X_n$ are independent. What is $\min(X_1, \ldots, X_n)$? $\max(X_1, \ldots, X_n)$?

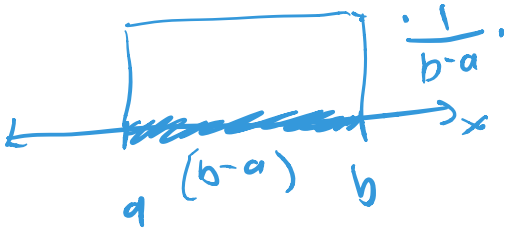
- [Sp17] Q7 part 3 (similar to darts)
- How to use testHalt (or how something halts in general?)

(Q7).



$f_{X,Y}(x,y) = 1$ if (x,y) shaded.

$f_{X,Y}(x,y)$ uniform



$\mathbb{P}[X \geq x]$.

c) $\mathbb{E}[Y]$?

$$f_Y(y) = \int_{-y}^{y} f_{X,Y}(x,y)\, dx.$$

$$= \int_{-y}^{y} 1\, dx$$

$$= x\Big|_{-y}^{y} = 2y. \quad \text{on } [0,1].$$

$$\mathbb{E}[Y] = \int_0^1 y \cdot f_Y(y)\, dy$$

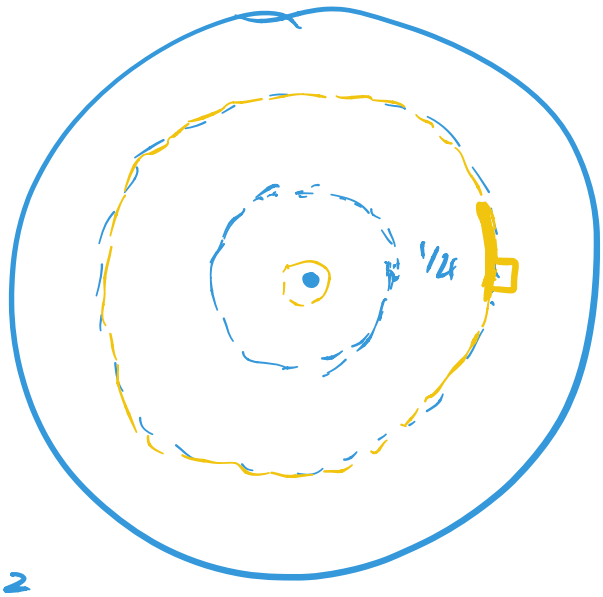
$$= \int_0^1 2y^2\, dy = 2\frac{y^3}{3}\Big|_0^1$$

$$= \boxed{\frac{2}{3}}.$$

Darts.

ok - uniform over area

good - uniform over distance from center.



a) $f_X(x) = 2x$

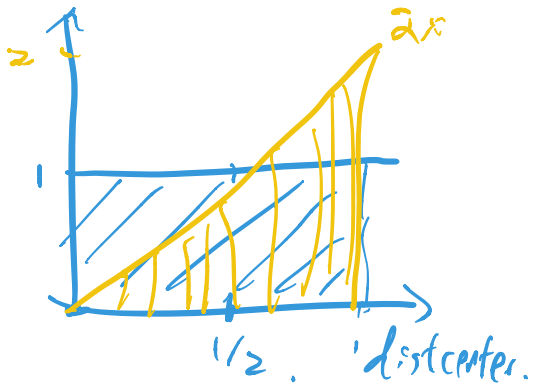
L_2 dist from center.

$$F_X(x) = \mathbb{P}(X \leq x) = \frac{x^2\pi}{\pi} = x^2.$$

$f_X(x) = \frac{dF}{dx} = 2x$ ok

of x

Y - uniform



$f_Y(y) = 1$ good.

$f_Z(z) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (2x)$?

b). $P(\text{good} \mid X = (\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}))$

$\sqrt{(\frac{1}{2\sqrt{2}})^2 + (\frac{1}{2\sqrt{2}})^2}$ $c^2 = a^2 + b^2$

$= \sqrt{\frac{1}{8} + \frac{1}{8}} = \sqrt{1/4}$

$P(\text{good} \mid X \in (\frac{1}{4}, \frac{1}{4} + \delta)) = \frac{P(\text{good} \cap X \in (\frac{1}{4}, \frac{1}{4} + \delta))}{P(\text{good} \cap \quad) + P(\text{bad} \cap \quad)}$

$= \frac{1/2\ \delta}{\quad} = \frac{1}{2} +$