Dis 12B : Applications.

Warm-up: (from last dis)

↑ 10

♡ 100

3 Working with the Law of Large Numbers

(a) A fair coin is tossed multiple times and you win a prize if there are more than 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

10

(b) A fair coin is tossed multiple times and you win a prize if there are more than 40% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

100

(c) A fair coin is tossed multiple times and you win a prize if there are between 40% and 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

100

(d) A fair coin is tossed multiple times and you win a prize if there are exactly 50% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

10.

1 Planetary Party

(a) Suppose we are at party on a planet where every year is 2849 days. If 30 people attend this party, what is the exact probability that two people will share the same birthday? You may leave your answer as an unevaluated expression.

$\mathbb{P}(\text{no collisions}) = \left(1-\frac{1}{2849}\right)\left(1-\frac{2}{2849}\right)\left(1-\frac{3}{2849}\right)\cdots\left(1-\frac{29}{2849}\right)$

$\mathbb{P}(\text{share}) = 1-$ ↑

(b) From lecture, we know that given n bins and m balls, $\mathbb{P}[\text{no collision}] \approx \exp(-m^2/(2n))$. Using this, give an approximation for the probability in part (a).

$\mathbb{P}(\text{no collison}) = e^{(-30^2/2\cdot 2849)} \approx 0.854$

$\mathbb{P}(\text{collision}) = 0.146.$

(c) What is the minimum number of people that need to attend this party to ensure that the probability that any two people share a birthday is at least 0.5? You can use the approximation you used in the previous part.

$e^{-m^2/2\cdot 2849} < 0.5 \Leftrightarrow \frac{-m^2}{2\cdot 2849} < \ln(0.5)$

$m > \sqrt{-2\ln(0.5)\cdot 2849} \approx 62.8$

$\boxed{m \geq 63}$

(d) Now suppose that 70 people attend this party. What the is probability that none of these 70 individuals have the same birthday? You can use the approximation you used in the previous parts.

$e^{-70^2/2\cdot 2849} \approx \underline{0.423}$

3 The Memoryless Property

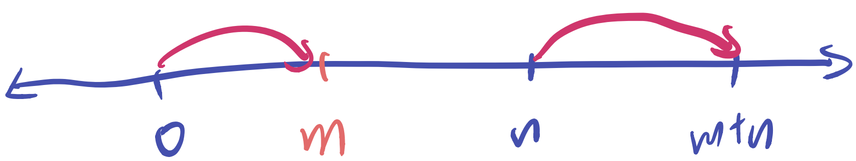
pos. int

Let X be a discrete random variable which takes on values in $\mathbb{Z}_+$. Suppose that for all $m,n \in \mathbb{N}$, we have $\mathbb{P}(X > m+n \mid X > n) = \mathbb{P}(X > m)$. Prove that X is a geometric distribution. Hint: In order to prove that X is geometric, it suffices to prove that there exists a $p \in [0,1]$ such that $\mathbb{P}(X > i) = (1-p)^i$ for all $i > 0$.

$$\boxed{\mathbb{P}(X > m+n \mid X > n) = \mathbb{P}(X > m)}$$

"memoryless property"

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$



$$\frac{\mathbb{P}(X > m+n \cap X > n)}{\mathbb{P}(X > n)} = \frac{\mathbb{P}(X > m+n)}{\mathbb{P}(X > n)} = \mathbb{P}(X > m)$$

$$\Rightarrow \mathbb{P}(X > m+n) = \mathbb{P}(X > m)\,\mathbb{P}(X > n)$$

$$\underbrace{\mathbb{P}(X > i) = (1-p)^i} \qquad \mathbb{P}(X > 5) = \mathbb{P}(X > 2)\,\mathbb{P}(X > 3)$$

$$\mathbb{P}(X > i) = \mathbb{P}(X > 1)\,\mathbb{P}(X > 1)\,\mathbb{P}(X > i-2)$$

$$= \underbrace{\mathbb{P}(X > 1)\,\mathbb{P}(X > 1)\cdots\mathbb{P}(X > 1)}_{i}$$

$$= \mathbb{P}(X > 1)^i$$

$$1-p = \mathbb{P}(X > 1) \Rightarrow \boxed{p = \mathbb{P}(X = 1)}$$