Dis 10B: Random Variables

Puzzle of the Day

Alice and Bob are playing a game. They are teammates, so they will win or lose together. Before the game starts, they can talk to each other and agree on a strategy.

When the game starts, Alice and Bob go into separate soundproof rooms – they cannot communicate with each other in any way. They each flip a coin and note whether it came up Heads or Tails. (No funny business allowed – it has to be an honest coin flip and they have to tell the truth later about how it came out.) Now Alice writes down a guess as to the result of Bob's coin flip; and Bob likewise writes down a guess as to Alice's flip.

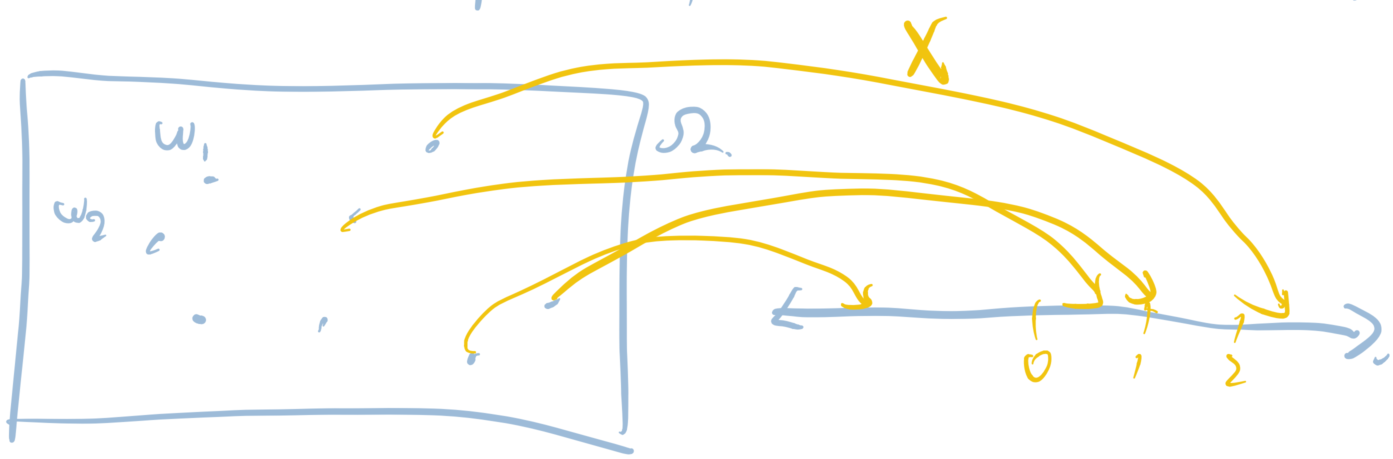
If either or both of the written-down guesses turns out to be correct, then Alice and Bob both win as a team. But if both written-down guesses are wrong, then they both lose.

The puzzle is this: Can you think of a strategy Alice and Bob can use that is guaranteed to win every time?

Review

A <u>random variable</u> X on a sample space Ω assigns every sample ω a real number $X(\omega)$.

A <u>distribution</u> of a r.v. X is its values and their associated probabilities, i.e. $\{(a, \mathbb{P}(X=a)) \mid a \in \mathcal{A}\}$.



Bin(n,p) = flip n coins, heads w/ probability p

1 Pullout Balls

Suppose you have a bag containing six balls numbered $1, 2, 3, 4, 5, 6$.

(a) You perform the following experiment: pull out a single ball and record its number. What is the expected value of the number that you record?

(b) You repeat the experiment from part (a), except this time you pull out two balls together and record the product of their numbers. What is the expected value of the total that you record?

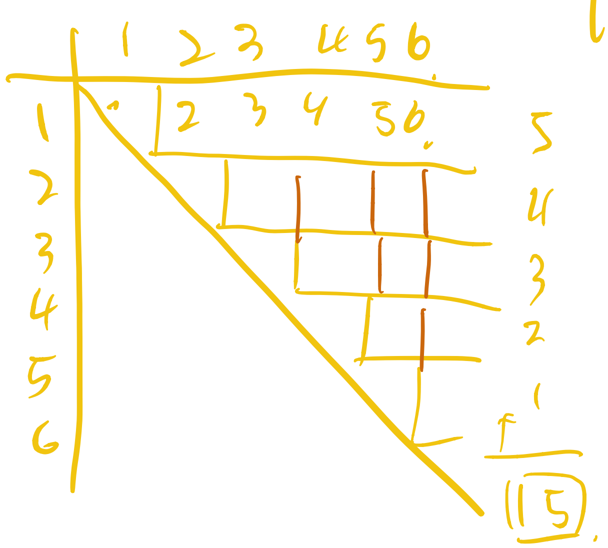
(a) $X =$ number I pull out

$$\mathbb{E}[X] = \sum_{x \in \Omega} x \cdot \mathbb{P}(X = x) = \sum_{k=1}^{6} k \cdot \mathbb{P}(X = k).$$

$$= \frac{1}{6} \sum_{k=1}^{6} k = \frac{1}{6} \cdot \frac{6 \cdot 7}{2} = 3.5.$$

(b). $Y =$ product of the two #s.

$$\mathbb{E}[Y] \overset{?}{=} \frac{\sum_{i=1}^{6} \left(i \times \sum_{j=i+1}^{6} j \right)_{j=i}}{\binom{6}{2}} = \frac{1 \cdot 20 + 2 \cdot 18 + 3 \cdot 15 + 4 \cdot 11 + 5 \cdot 6}{15}$$



$$= \boxed{\frac{35}{3}}$$

X, Y ind

$\Rightarrow \quad \mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$

2 How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let X denote the number of queens you draw.

(a) What is $\mathbb{P}(X=0)$, $\mathbb{P}(X=1)$, $\mathbb{P}(X=2)$ and $\mathbb{P}(X=3)$?

AAB 3 3

(b) What do your answers you computed in part a add up to?

ABC 3 6.

(c) Compute $\mathbb{E}(X)$ from the definition of expectation.

(d) Are the X_i indicators independent?

(a) $\mathbb{P}(X=0) = \binom{48}{3} / \binom{52}{3} = \frac{48 \cdot 47 \cdot 46}{52 \cdot 51 \cdot 50} = \frac{4324}{5525}$ 0

$\mathbb{P}(X=1) = \binom{4}{1}\binom{48}{2} / \binom{52}{3} = \frac{1128}{5525}$

$\mathbb{P}(X=2) = \binom{4}{2}\binom{48}{1} / \binom{52}{3} = \frac{72}{5525}$

$\mathbb{P}(X=3) = \binom{4}{3} / \binom{52}{3} = \frac{1}{5525}$.

(b) 1.

(c) $\mathbb{E}[X] = 0 \cdot \frac{4324}{5525} + 1 \cdot \frac{1128}{5525} + 2 \cdot \frac{72}{5525} + 3 \cdot \frac{1}{5525}$
$= 3/13$.

(d) $X_1 = 1$ if card 1 is Q $\qquad \mathbb{P}(X_2 \overset{=1}{} \mid X_1 = 1) \neq \mathbb{P}(X_2 = 1)$.

$X_2 = 2$ " "

$X_3 =$ " "

indicator r.v. : r.v. which is either 0/1, i.e. (Bernoulli)

$$X = \begin{cases} 1 & \text{w/ prob } p \\ 0 & \text{w/ prob } 1-p. \end{cases}$$

prop: $\mathbb{E}[X] = 1 \cdot p + 0 \cdot (1-p) = p.$

H T H H T T

X_1 = 1 if coin 1 comes up H

X_2 = " 2 "

⋮

X_6 = " 6 ".

$$\mathbb{E}[X] = \mathbb{E}[X_1 + \cdots + X_6]$$

X = # of heads

$$= \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_6].$$

$$= 6 \cdot \mathbb{E}[X_1]$$

$$= 6 \cdot \mathbb{P}(X_1 = 1).$$

$$= 6 \cdot p.$$

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Alice - guess the opposite

Bob - guess his coin.

Bob

Alice	H	T
H	A: T ✓ Bob: H	A: T ✓ Bob: T
T	A: H ✓ B: T ✓	A: H B: T ✓

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3 Head Count

Consider a coin with $\mathbb{P}(\text{Heads}) = 2/5$. Suppose you flip the coin 20 times, and define X to be the number of heads.

(a) Name the distribution of X and what its parameters are.

$\text{Bin}(n,p)$, $n = \#$ of coins/flips, $p = \mathbb{P}(\text{Heads})$ $\quad \text{Bin}(20, \frac{2}{5})$.

1/2

(b) What is $\mathbb{P}(X = 7)$?

$$\mathbb{P}(X=7) = \binom{20}{7}\left(\frac{2}{5}\right)^7\left(\frac{3}{5}\right)^{13}.$$

1/2

(c) What is $\mathbb{P}(X \geq 1)$? Hint: You should be able to do this without a summation.

$$1 - \mathbb{P}(X=0) = 1 - \left(\frac{3}{5}\right)^{20}.$$

(d) What is $\mathbb{P}(12 \leq X \leq 14)$?

$$\mathbb{P}(12 \leq X \leq 14) = \mathbb{P}(X=12) + \mathbb{P}(X=13) + \mathbb{P}(X=14)$$

$$=$$