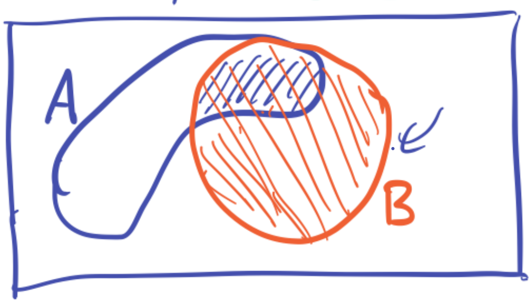
Dis 10A: Multiple Events

Review

Conditional Probability: $\mathbb{P}(A \mid B)$, read "prob. of A given B"



there's alot of related concepts in conditional probability.

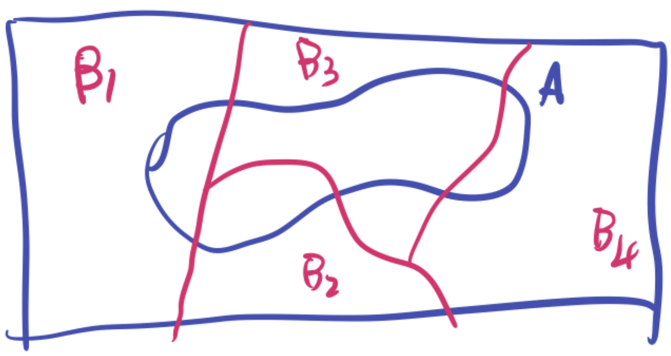
Product Rule: $\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B) / \mathbb{P}(B)$ (self-evident)

↳ sometimes seen as $\mathbb{P}(A \mid B) \cdot \mathbb{P}(B) = \mathbb{P}(A \cap B)$.

Bayes' Rule:
$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)}.$$

Law of Total Probability: If $B_1, B_2, \ldots, B_n$ partition the space,

$$\mathbb{P}(A) = \mathbb{P}(A \mid B_1)\,\mathbb{P}(B_1) + \mathbb{P}(A \mid B_2)\,\mathbb{P}(B_2) + \cdots + \mathbb{P}(A \mid B_n)\,\mathbb{P}(B_n)$$



1 Probability Potpourri

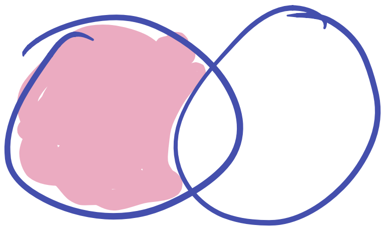
Prove a brief justification for each part.

(a) For two events A and B in any probability space, show that $\mathbb{P}(A \setminus B) \geq \mathbb{P}(A) - \mathbb{P}(B)$.

(b) If $|\Omega| = n$, how many distinct events does the probability space have?

(c) Suppose $\mathbb{P}(D \mid C) = \mathbb{P}(D \mid \overline{C})$, where $\overline{C}$ is the complement of C. Prove that D is independent of C.

(a) A



B

$A \setminus B = A - A \cap B.$

≤ 1

$$\mathbb{P}(A \cap B) = \mathbb{P}(B) \cdot \mathbb{P}(A \mid B) \leq \mathbb{P}(B)$$

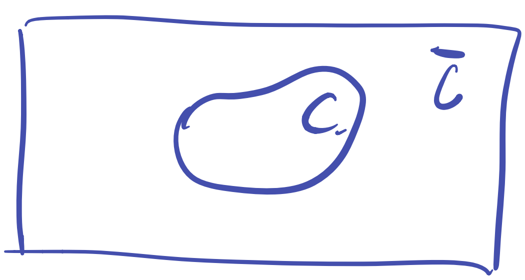
$$\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$$

$$\geq \mathbb{P}(A) - \mathbb{P}(B).$$

(b) $A \subseteq \Omega \Rightarrow$ how many subsets of Ω are there?

$$|\mathcal{P}(\Omega)| = 2^n.$$

(c).



$$\mathbb{P}(D) = \mathbb{P}(D \mid C) \cdot \mathbb{P}(C) + \mathbb{P}(D \mid \bar{C}) \cdot \mathbb{P}(\bar{C})$$

$$= \mathbb{P}(D \mid C) \cdot \mathbb{P}(C) + \mathbb{P}(D \mid C) \cdot \mathbb{P}(\bar{C})$$

$$= \mathbb{P}(D \mid C)[\mathbb{P}(C) + \mathbb{P}(\bar{C})] = \mathbb{P}(D \mid C)$$

2 Aces

Consider a standard 52-card deck of cards:

(a) Find the probability of getting an ace or a red card, when drawing a single card.

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

$(b_1, b_2, b_3, b_4 \ldots)$.

b_i = bin # of the ith ball

3 Balls and Bins

Throw n balls into n labeled bins one at a time.

(a) What is the probability that the first bin is empty?

$$\left(\frac{n-1}{n}\right)^n$$

(b) What is the probability that the first k bins are empty?

$$\left(\frac{n-k}{n}\right)^n$$

(c) Let A be the event that at least k bins are empty. Notice that there are $m = \binom{n}{k}$ sets of k bins out of the total n bins. If we assune A_i is the event that the i^{th} set of k bins is empty. Then we can write A as the union of A_i's.

$$A = \bigcup_{i=1}^{m} A_i.$$

Write the union bound for the probability A.

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{i=1}^{m} A_i\right) \le \sum_{i=1}^{m} \mathbb{P}(A_i)$$

(d) Use the union bound to give an upper bound on the probability A from part (c).

$$\le m \cdot \left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n.$$

(e) What is the probability that the second bin is empty given that the first one is empty?

B_1: bin 1 empty
B_2: bin 2 empty

$$\mathbb{P}(B_2 \mid B_1) = \mathbb{P}(B_1 \cap B_2) / \mathbb{P}(B_1) = \left(\frac{n-2}{n}\right)^n \Big/ \left(\frac{n-1}{n}\right)^n = \left(\frac{n-2}{n-1}\right)^n$$

(f) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

No. second event => first w/ prob 1.

(g) Are the events that "the first bin is empty" and "the second bin is empty" independent?

$$\mathbb{P}(B_2 \mid B_1) = \left(\frac{n-2}{n-1}\right)^n \neq \mathbb{P}(B_2) = \left(\frac{n-1}{n}\right)^n.$$