CS 70 Discussion #3 Notes: Graph Theory

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These are some notes I typed up attempting to explain some questions on Discussion #3 in more detail, and a brief note on where planarity bounds come from.

1 Banquet Arrangement

Here's Problem 1 from the discussion worksheet.

Problem 1. Suppose n people are attending a banquet, and each of them has at least m friends ($2 \leq m \leq n$), where friendship is mutual. Prove that we can put at least $m+1$ of the attendants on the same round table, so that each person sits next to his or her friends on both sides.

Before I start, I'd also like to thank Dan for showing me how to finish the proof with this approach, which I find more motivated.

Solution. As an example, suppose Alice, Bob, Claire, and Dom are attending, and the friend pairings are (Alice, Claire), (Alice, Dom), (Bob, Claire), (Bob, Dom), so $n = 4$ and $m = 2$ in this case. We actually can't choose exactly three of them so that they all sit next to their friends, but we can certainly choose to sit the four them in the order (Alice, Claire, Bob, Dom).

If you haven't realized already, it's helpful to recast this problem in terms of graphs to solve it in general. A big tipoff is both the fact that there are people and mutual friendships which strongly hint to being vertices and edges respectively as well as the topic of this discussion. So we will set

$$\text{people} \mapsto \text{vertices}$$

and

$$\text{friendship} \mapsto \text{edges}\ .$$

If we were to represent the above problem in graphs, it'd look something like this:

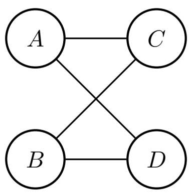

Figure 1: Casting the problem in terms of graphs.

We also see that sitting people around a table so that they sit next to their friends is just asking to find a cycle of length at least $m+1$. Using the above graph, it's easy to find one such cycle, $ACBD$, by starting at A and walking on the graph until you come back to A.

So that's the easy part of the problem. Now let's see how we would create such a cycle. Naively, we might start at an arbitrary vertex, say v_0, and try to walk along this graph. We know v_0 has at least m neighbors, so let's visit one of them, v_1.

Now v_1 also has at least m neighbors, but we've already been to one of them, v_0, so it really only has $m-1$ unvisited neighbors, so let's visit one of them, v_2. We can keep repeating this process of visiting unvisited neighbors until we're out of new neighbors to get a path

$$P = v_0 v_1 \dots v_l.$$

One thing we should note is that in the worst case, the number of unvisited neighbors for each v_i is at least $m-i$, which is if all of the i vertices already in the path are neighbors of v_i. So $l \geq m$, or else we would still have neighbors we could visit.

All that's left for us now is to get a cycle from this path. Since we're stuck when we reach v_l, all of its neighbors must be in P. So let's make a new path starting at v_l by backpedaling from v_l until we reach the last neighbor of v_l in P, say v_k. Now we have a path

$$P' = v_l v_{l-1} \dots v_k$$

where P' is also at least $m+1$ since v_l and all of its neighbors are in the path. But v_k is neighbors with v_l, so this is also a cycle, and we're done. □

We used the principle of trying things until we get stuck, and trying to fix things until we're free. You'll find that this often works surprisingly well.

The official solution is quite slick as well, but less motivated. It considers the *longest* path in the graph, and constructs the cycle from that. This is an example of the **Extremal Principle**, where one looks at either the largest or the smallest object satisfying some property.

Official Solution. Let $P = v_0 v_1 \dots v_l$ be a longest path in the graph. Such a path exists because the length of paths is bounded above by n. All neighbors of v_0 must be in P, since otherwise P can be extended to be even longer by appending this edge at the beginning of path P. Let k be the maximum index of neighbors of v_0 along P. Since v_0 has at least m neighbors, we must have $k \geq m$. Then $v_0 v_1 \dots v_k v_0$ gives us the desired cycle. □

Here's a cute example of the Extremal Principle applied in another setting. Before you read on, think about it for a few minutes.

Problem 2. There are n students standing in a field such that the distance between each pair is distinct. Each student is holding a ball, and when the teacher blows a whistle, each student throws their ball to the nearest student. Prove that there is a pair of students that throw their balls to each other.

Proof. Consider the two students closest to each other. Since they are each other's nearest student, they must throw their balls to each other. □

Try this one for another extremal flavored graph problem. How would you do it without looking at the extreme cases?

Exercise 1.1. Show that every tree G contians at least two leaves, or vertices of degree 1.

2 Induction on Edges and Vertices

Problem 2 from the discussion also gave people some trouble, so I'll try to explain it some more in depth. Here is the problem, paraphrased.

Problem 3. An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors.

(a) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors.

(b) Show that any tree with maximum degree $d \geq 1$ can be edge colored with d colors.

The solution is to use induction on the number of edges in the first question, and induction on the number of vertices in the second. I'm not going to re-explain the solution (you can find it online) but I'll try to explain why it's fine to induct this way.

Imagine if I had the following problem:

Question. Prove that any graph with maximum degree 4 can be edge colored with 7 colors.

Now induction, especially on the number of edges, seems like a viable approach. We're proving some statement about all graphs, so let's assume its true for a graph with m edges and prove it for one with $m + 1$ edges (both with max degree 4). What about this problem:

Question. Prove that any graph with maximum degree 10 can be edge colored with 19 colors.

See what I'm getting at? Just because we have a variable d in our original statement doesn't mean we need to induct over d. We can treat d as a constant and show the statement is true for all graphs given a specific d, which then shows the statement is true for all d.

Another analogy: we don't need to induct over our variable d much like how the number of vertices and edges are also variables, but we don't need to induct over both.

3 Planarity Bounds

Recall that for any planar, connected graph G, if G has V vertices, E edges, and F faces, Euler's formula tells us that

$$V - E + F = 2.$$

We can derive many useful bounds using this formula that involve only two of the three quantities, which is helpful especially for showing that certain graphs are nonplanar. Usually we omit faces since those are tricky to quantify, so let's try to compare the number of faces to the number of edges.

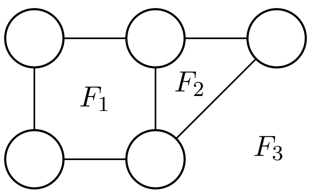

Figure 2: An example planar graph G_1 with its three faces marked.

Let's look at the graph G_1 shown in Figure 2. Notice that every edge is a part of exactly two faces. So we can count the number of edges also by looking at how many edges border each face. Let $|F_i|$ denote the number of edges that border the face F_i. Then,

$$\begin{aligned} 2E &= |F_1| + |F_2| + |F_3| \\ &= 4 + 3 + 5 \\ &\geq 3 + 3 + 3 \\ &= 3F \end{aligned}$$

where we used the very deep fact that every face is bordered by at least 3 edges (how else do you have a face?). Hence, we've arrived at the inequality $2E \geq 3F$. Now if we solve our formula for F and substitute, we get the all important bound

$$2 - V + E = F \leq \frac{2}{3}E \implies \boxed{E \leq 3V - 6}.$$

One extension of this is to find a bound when G is triangle-free, i.e. no face is bounded by 3 edges. I'll leave that as an exercise for you.

Exercise 3.1. Show that if G is a connected, planar, triangle-free graph, then $E \leq 2V - 4$. Use this to show that $K_{3,3}$ is nonplanar.

4 Challenge Question

Problem 4 (HW #3.7)**.** Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G , there are at least two edges among them. Prove that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Hint: Proof by contradiction when $v = 7$. What do we know about groups of five vertices in a planar graph?