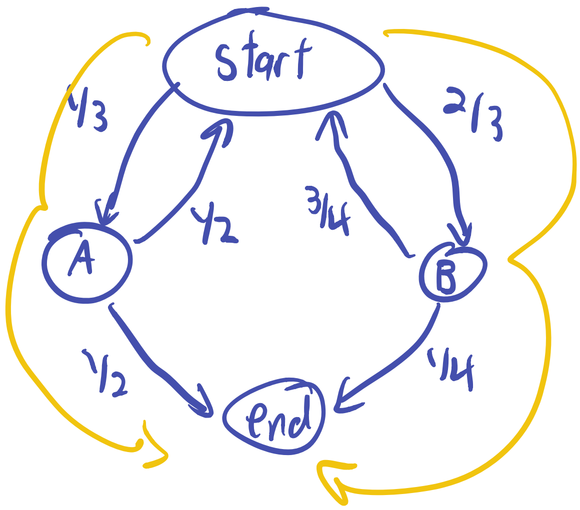
Dis 14B: Markov Chains

- Last discussion! ☹.

Cute problem: Find (w/o paper) the expected number of steps it takes to go from start to end.



$\frac{1}{3}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{1}{4}=\frac{1}{3}$.

Review

A Markov Chain's defining property is the amnesic property "Markov Property",

i.e.

$$\mathbb{P}[X_n = x_n \mid X_{n-1} = x_{n-1}, \ldots, X_0 = x_0] = \mathbb{P}[X_n = x_n \mid X_{n-1} = x_{n-1}],$$

→ let's us encapsulate all the info about a MC into a transition matrix.

Q: What if this probability relied on the last two states?

☆ P is the transition matrix

☆ π_0 is the initial distribution

☆ $\mathcal{X}$ is the state space

☆ $\pi_n = \pi_0 P^n$. ←

Key Properties + Concepts

(stationary)
Invariant distribution: π s.t. $\pi = \pi P$ (balance equations)
↳ convergence, steady-state, etc.

A MC is <u>irreducible</u> if it can reach every state j from every state i.
↳ i.e. connected as a directed graph.

A MC is <u>aperiodic</u> if every state has $d(i)=1$, where $d(i)$ is the gcd of all of its "periods".
↳ note in an irreducible MC that all states have the same $d(i)$ since they are all "reachable" from each other.

BIG.
<u>Theorem</u> If a finite MC is irreducible, it has a unique stationary distr. Furthermore, if that MC is also aperiodic, then every initial distr. converges to the stationary distr.

Hitting Times: finding $\beta(i)$, i.e. expected # of steps to reach a fixed state from state i

↳ <u>First Step Equations</u>: to find time to hit some set of states A,

$$\beta(i) = 0 \quad \text{if } i \in A$$

$$\beta(i) = 1 + \sum_{j \in \mathcal{X}} P(i,j)\,\beta(j)$$

← "one-step" into the future.

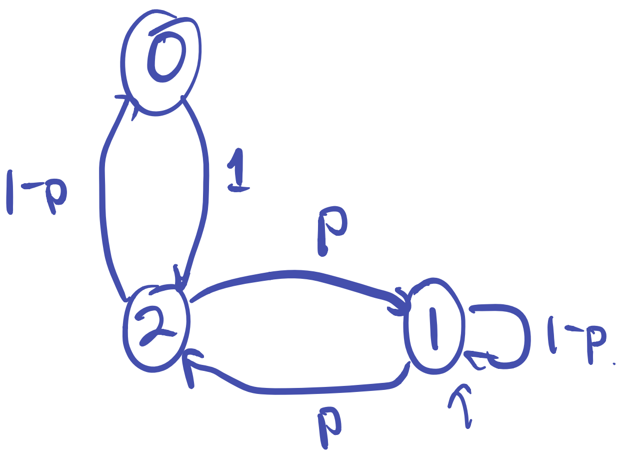
Probability of Hitting A before B: same idea, but $\alpha(i) = \sum_{j \in \mathcal{X}} P(i,j)\,\alpha(j)$.

3 Allen's Umbrella Setup

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

(a) Model this as a Markov chain. What is $\mathscr{X}$? Write down the transition matrix.

$\mathcal{X}$: # of umbrellas at curr loc. $= \{0, 1, 2\}$.



$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & & & \\ 1 & & & \\ 2 & & & \end{array} \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}$$

(b) What is the transition matrix after 2 trips? n trips? Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

P^2, P^n. irreducible ✓ aperiodic ✓

inv dist: $\pi = \pi P$. $\Rightarrow$ $\pi(P - I) = 0$.

$\pi = [\pi(0) \quad \pi(1) \quad \pi(2)]$.

$$\Rightarrow \quad [\pi(0) \quad \pi(1) \quad \pi(2)] \begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & p \\ 1-p & p & -1 \end{bmatrix} = 0.$$

trick: replace last col w/ $\pi(0) + \pi(1) + \pi(2) = 1$.

$$[\pi(0) \quad \pi(1) \quad \pi(2)] \begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & 1 \\ 1-p & p & 1 \end{bmatrix} = [0 \quad 0 \quad 1].$$

$$\Rightarrow \begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix} = \frac{1}{3-p} \begin{bmatrix} \underline{1-p} & 1 & 1 \end{bmatrix}.$$

fraction of time w/ no umbrella in the rain:

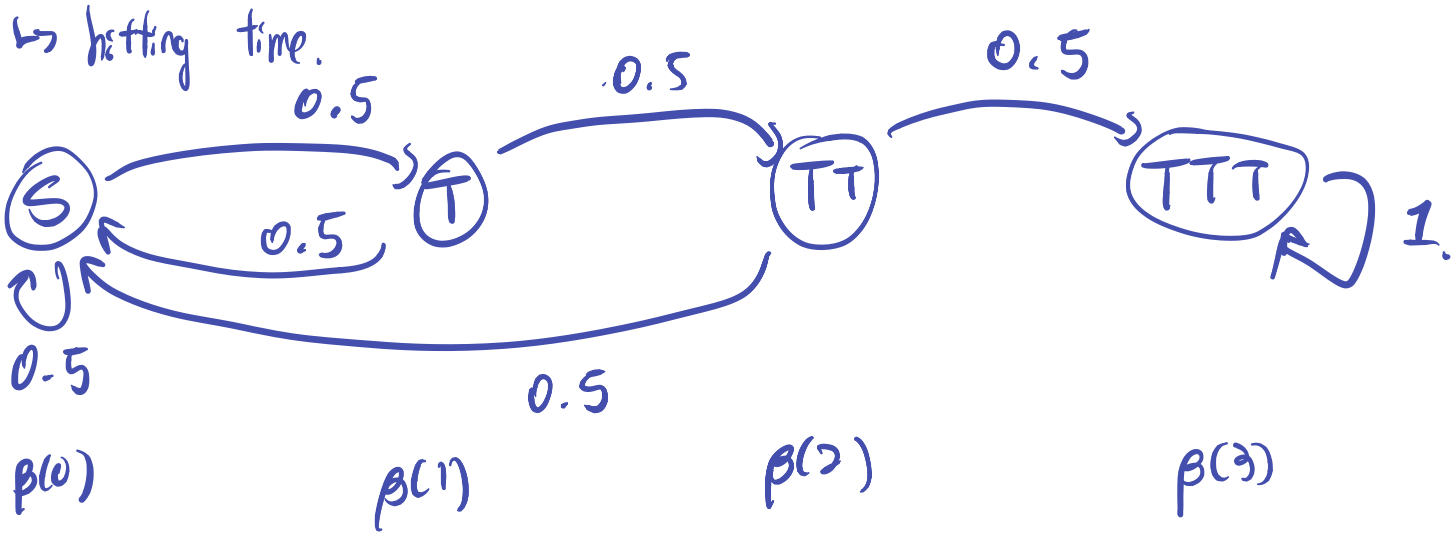
$$\frac{1-p}{3-p} \cdot p = \boxed{\frac{p(1-p)}{3-p}}.$$

4 Three Tails

You flip a fair coin until you see three tails in a row. What is the average number of heads that you'll see until getting TTT?

Hint: How is this different than the number of *coins* flipped until getting TTT?

↳ hitting time.



$\beta(i)$ = # of step exp. to reach TTT.

$$\beta(2)=\frac{1}{2}\left(\beta(3)+\cancel{1}\right) \quad T$$
$$+\frac{1}{2}\left(\beta(0)+1\right) \quad H$$

$$\beta(3) = 0$$
$$\beta(2)=\frac{1}{2}+\frac{1}{2}\beta(3)+\frac{1}{2}\beta(0) \quad ✓.$$
$$\beta(1)=\frac{1}{2}+\frac{1}{2}\beta(2)+\frac{1}{2}\beta(0)$$
$$\beta(0)=\frac{1}{2}+\frac{1}{2}\beta(1)+\frac{1}{2}\beta(0). \quad ✓.$$
$$\Downarrow$$
$$\beta(0)=1+\beta(1).$$
$$\Downarrow$$
$$\beta(1)=\frac{1}{2}+\frac{1}{2}\beta(2)+\frac{1}{2}(1+\beta(1)) \Rightarrow \beta(1)=2+\beta(2).$$
$$\Downarrow$$
$$\beta(2)=\frac{1}{2}+\frac{1}{2}\beta(3)+\frac{1}{2}(3+\beta(2)) \Rightarrow \beta(2)=4+\beta(3). \quad ✓.$$
$$=4.$$
$$\beta(2)=4 \Rightarrow \beta(1)=6 \Rightarrow \underline{\beta(0)=7.}$$

$$\beta(i) = 1 + \sum_{j \in X} p(i,j)\,\beta(j)$$

$$= \sum_{j \in X} p(i,j)\left[\beta(j) + 1\right].$$