Dis 09A: Intro to Discrete Probability

- no new announcements; do your homework + vitamins!

Review

A probability space is two things:

1) A sample space Ω, and
2) A probability function $\mathbb{P}(\omega)$ which assigns a "probability" to each sample pt. $\omega \in \Omega$.

They satisfy 2 axioms:

- $\mathbb{P}(\omega) \geq 0$ for all $\omega \in \Omega$
- $\sum_{\omega} \mathbb{P}(\omega) = 1$

(Referred to as $(\Omega, \mathbb{P})$).

ex: Rolling a Die

Ω = outcomes of a die $\mathbb{P}$: fair die

$\mathbb{P}(1) = 1/6$	$\mathbb{P}(2) = 1/6$	$\mathbb{P}(3) = 1/6$
1	2	3
$\mathbb{P}(4) = 1/6$	$\mathbb{P}(5) = 1/6$	$\mathbb{P}(6) = 1/6$.
4	5	6

Ω = outcomes of a die $\mathbb{P}$ = biased to 1

$\mathbb{P}(1) = 1/2$	$\mathbb{P}(2) = 1/10$	$\mathbb{P}(3) = 1/10$
1	2	3
$\mathbb{P}(4) = 1/10$	$\mathbb{P}(5) = 1/10$	$\mathbb{P}(6) = 1/10$
4	5	6

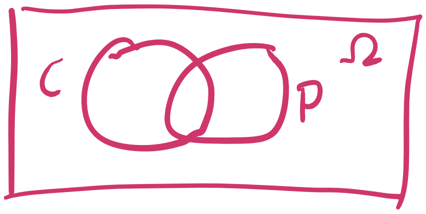
Def An event in a prob. space $(\Omega, \mathbb{P})$ is a subset $A \subseteq \Omega$.

we define $\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega)$

1 Venn Diagram

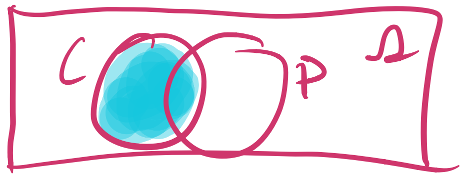
Out of 1,000 computer science students, 400 belong to a club (and may work part time), 500 work part time (and may belong to a club), and 50 belong to a club and work part time.

(a) Suppose we choose a student uniformly at random. Let C be the event that the student belongs to a club and P the event that the student works part time. Draw a picture of the sample space Ω and the events C and P.



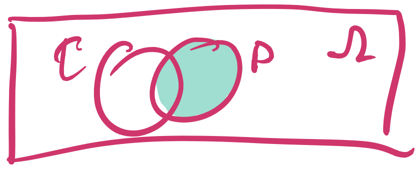
(b) What is the probability that the student belongs to a club?

$\mathbb{P}(C) = \frac{400}{1000} = \frac{2}{5}$



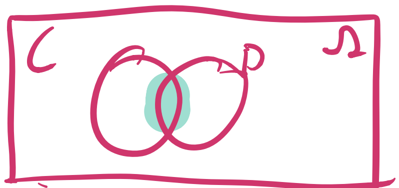
(c) What is the probability that the student works part time?

$\mathbb{P}(P) = \frac{500}{1000} = \frac{1}{2}$



(d) What is the probability that the student belongs to a club AND works part time?

$\mathbb{P}(C \cap P) = \frac{50}{1000} = \frac{1}{20}$



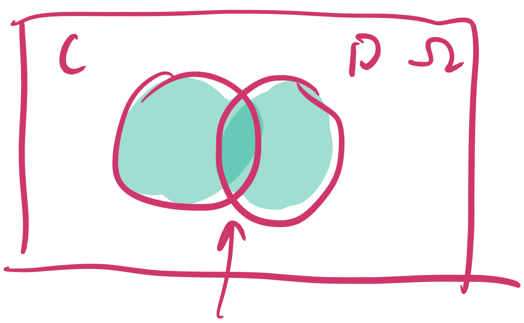
(e) What is the probability that the student belongs to a club OR works part time?

$\mathbb{P}(C \cup P)$

$= \mathbb{P}(C) + \mathbb{P}(P)$

$- \mathbb{P}(C \cap P)$

$= \frac{2}{5} + \frac{1}{2} - \frac{1}{20} = \frac{17}{20}.$



2 Flippin' Coins

Suppose we have an unbiased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 1/2$ and probability of tails also $\mathbb{P}[T] = 1/2$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

$\Omega_1 = \{$ HHH, HTT, TTH, HHT, THH, TTT, HTH, THT, $\}$ $\quad \Omega_2 = \{H, T\}^3 \quad \Omega_3 = \{0, 1, 2, 3\}$.

1/8 3/8 3/8 1/8

{ HHT, HTH, THH }.

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

$\{$ THH, THT, TTH $\}$.

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

$A \cup B = \{$ TTT, HHT, HTH, THH $\}$.

(e) What is the probability of the outcome (H,H,T)?

1/8.

(f) What is the probability of the event that our outcome has exactly two heads?

$A = \{$ HHT, HTH, THH $\}$ $\quad |A| = 3$

$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega) = 3 \cdot \frac{1}{8} = 3/8$.

(g) What is the probability of the event that our outcome has at least one head?

complement: $\mathbb{P}$(0 heads) = 1/8. $(\{TTT\})$.

www: $1 - \frac{1}{8} = 7/8$

3 Counting & Probability

$x_i \geq 0$.

Consider the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 70$, where each x_i is a non-negative integer. We choose one of these solutions uniformly at random.

(a) What is the size of the sample space?

= how many sol's? => stars & bars

$= \text{total is } \binom{70+6-1}{6-1} = \boxed{\binom{75}{5}}$

(b) What is the probability that both $x_1 \geq 30$ and $x_2 \geq 30$?

$x_1' \geq 0$.

① $\underline{(x_1' + 30) + (x_2' + 30) + \dots + x_6 = 10}$ by placing 30 balls in x_1 and x_2.

$(x_1 - 30) + (x_2 - 30) + x_3 + \dots + x_6 = 10$

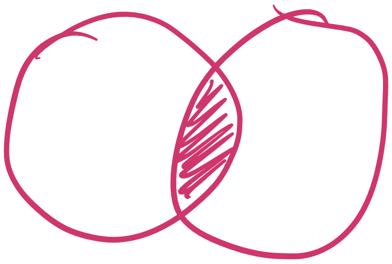
$x_1' = x_1 - 30$.

$x_1' \geq 0 \Rightarrow x_1 \geq 30$.

$\Rightarrow \binom{10+6-1}{6-1} = \binom{15}{5}$

$\boxed{\mathbb{P} = \binom{15}{5} / \binom{75}{5}}$

(c) What it the probability that either $x_1 \geq 30$ or $x_2 \geq 30$?



$A = \mathbb{P}(x_1 \geq 30) \rightarrow x_1' + x_2 + \dots + x_6 = 40$

$\Rightarrow \binom{40+6-1}{6-1}$

$= \binom{45}{5}$

$B = \mathbb{P}(x_2 \geq 30)$

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$= \boxed{\frac{\binom{45}{5}}{\binom{75}{5}} + \frac{\binom{45}{5}}{\binom{75}{5}} - \frac{\binom{15}{5}}{\binom{75}{5}}}$