Discussion 8A : Countability

Announcements

- HW8 is due this Friday at 10PM
- Vitamin 8 is due this Friday at 10PM
- MT1 Regrade Requests are due this Friday
- One-on-one sign-ups are due this Friday
- fill out a post MT1 feedback form : bit.ly/tyler-dis8a.

Our next example concerns the set of all binary strings (of any finite length), denoted $\{0,1\}^*$. Despite the fact that this set contains strings of unbounded length, it turns out to have the same cardinality as $\mathbb{N}$. To see this, we set up a direct bijection $f : \{0,1\}^* \to \mathbb{N}$ as follows. Note that it suffices to *enumerate* the elements of $\{0,1\}^*$ in such a way that each string appears exactly once in the list. We then get our bijection by setting $f(n)$ to be the nth string in the list. How do we enumerate the strings in $\{0,1\}^*$? Well, it's natural to list them in increasing order of length, and then (say) in *lexicographic* order (or, equivalently, numerically increasing order when viewed as binary numbers) within the strings of each length. This means that the list would look like

$$\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 1000, \ldots,$$

where ε denotes the empty string (the only string of length 0). It should be clear that this list contains each binary string once and only once, so we get a bijection with $\mathbb{N}$ as desired.

Our final countable example is the set of all polynomials with natural number coefficients, which we denote $\mathbb{N}(x)$. To see that this set is countable, we will make use of (a variant of) the previous example. Note first that, by essentially the same argument as for $\{0,1\}^*$, we can see that the set of all *ternary* strings $\{0,1,2\}^*$ (that is, strings over the alphabet $\{0,1,2\}$) is countable. To see that $\mathbb{N}(x)$ is countable, it therefore suffices to exhibit an injection $f : \mathbb{N}(x) \to \{0,1,2\}^*$, which in turn will give an injection from $\mathbb{N}(x)$ to $\mathbb{N}$. (It is obvious that there exists an injection from $\mathbb{N}$ to $\mathbb{N}(x)$, since each natural number n is itself trivially a polynomial, namely the constant polynomial n itself.)

How do we define f? Let's first consider an example, namely the polynomial $p(x) = 5x^5 + 2x^4 + 7x^3 + 4x + 6$. We can list the coefficients of $p(x)$ as follows: $(5,2,7,0,4,6)$. We can then write these coefficients as binary strings: $(101, 10, 111, 0, 100, 110)$. Now, we can construct a ternary string where a "2" is inserted as a separator between each binary coefficient (ignoring coefficients that are 0). Thus we map $p(x)$ to a ternary string as illustrated below:

It is easy to check that this is an injection, since the original polynomial can be uniquely recovered from this ternary string by simply reading off the coefficients between each successive pair of 2's. (Notice that this mapping $f : \mathbb{N}(x) \to \{0,1,2\}^*$ is not onto (and hence not a bijection) since many ternary strings will not be

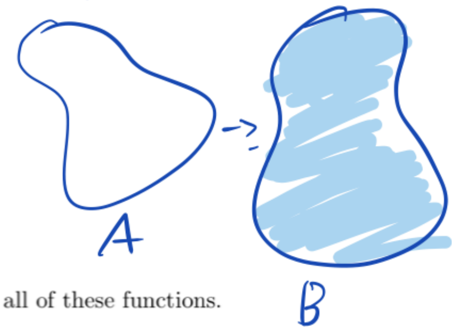
CCC - countable union of countable sets is countable.

$S_1, S_2, \ldots$ each countable. $\bigcup_{i=1}^{\infty} S_i$

Review

(from Dis 4A) Review (Bijections)

$f: A \to B$



- f is **onto** (surjective) if $\forall b \in B, \exists a \in A$ s.t. $f(a) = b$
  - i.e. every $b \in B$ has a pre-image.
- f is **one-to-one** (injective) if $\forall a, a' \in A, f(a) = f(a') \implies a = a'$.
  - i.e. different inputs map to different outputs,

one-to-one & onto = bijective.

Here's a helpful graphic illustrating the differences between all of these functions.

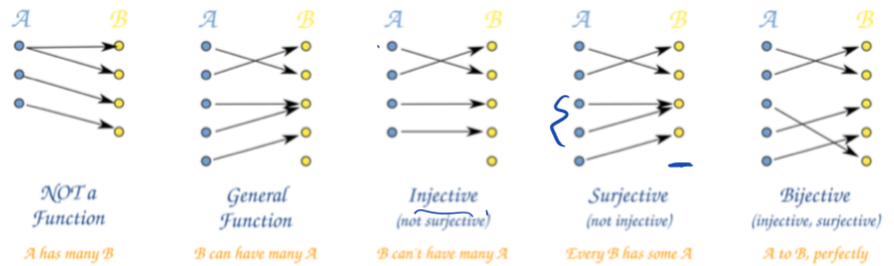


Figure 1: Examples of the four types of functions, and a non-function. Source: Math is Fun.

two sets A, B have the same cardinality (basically size) if $\exists$ a bijection $f: A \to B$

- (intuition) $f: A \to B$ onto means $|A| \geq |B|$ & $g: B \to A$ onto.
  $f: A \to B$ one-to-one means $|A| \leq |B|$.

S is countable if it is in bijection w/ $\mathbb{N}$ or some subset of $\mathbb{N}$.

Countable Sets

- $\mathbb{N}$, naturals
- $\mathbb{Z}$, $\mathbb{Q}$, ...
- $\mathbb{N} \times \mathbb{N}$
- finite set.

Uncountable Sets

- $\mathbb{R}$, real numbers
- infinite length binary strings
- $\mathcal{P}(\mathbb{N})$ = set of all subset of $\mathbb{N}$.

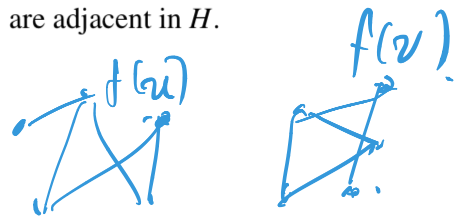
1 Graph Isomorphic

In graph theory, an isomorphism of graphs G and H is a bijection between the vertex sets of G and H

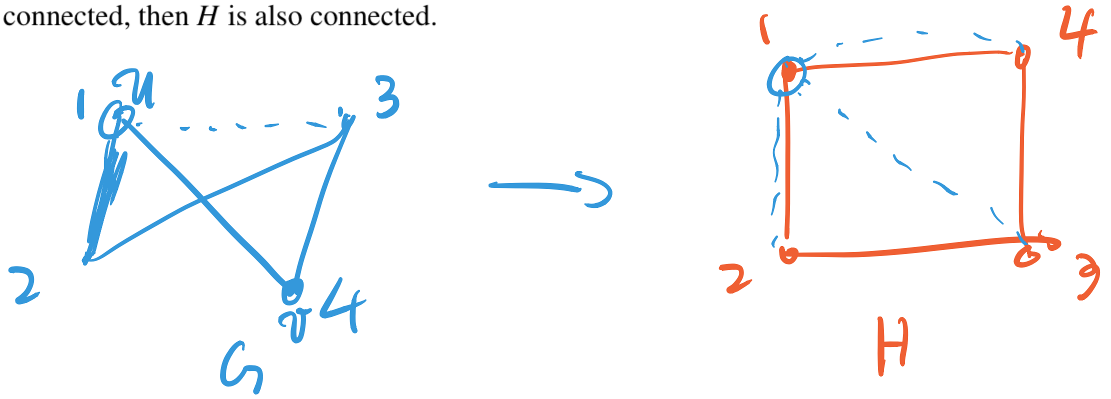
$$f : V(G) \to V(H)$$

such that any two vertices u, v of G are adjacent in G if and only if $f(u), f(v)$ are adjacent in H.

Prove the following:



1. The degrees of corresponding nodes $u, f(u)$ are the same.
2. There is a bijection between edges.
3. If G is connected, then H is also connected.



1. fix u, for any $v \in V(G)$, u, v are adjacent in G iff $f(u), f(v)$ are adjacent in H.

$\Rightarrow$ u, v are connected iff $f(u), f(v)$ are connected.

2. $(u, v) \in E(G)$,

$$(u, v) \longmapsto (f(u), f(v)).$$

3. Assume H is not connected. $\exists$ $f(u), f(v)$ that are not connected.

in G: $u \to x_1 \to x_2 \cdots \to x_n \to v$.

in H: $f(u) \to f(x_1) \to f(x_2) \cdots \to f(x_n) \to f(v)$. X

contradiction.

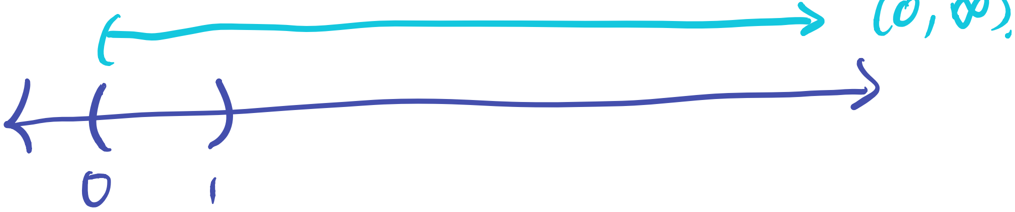
2 Countability Practice

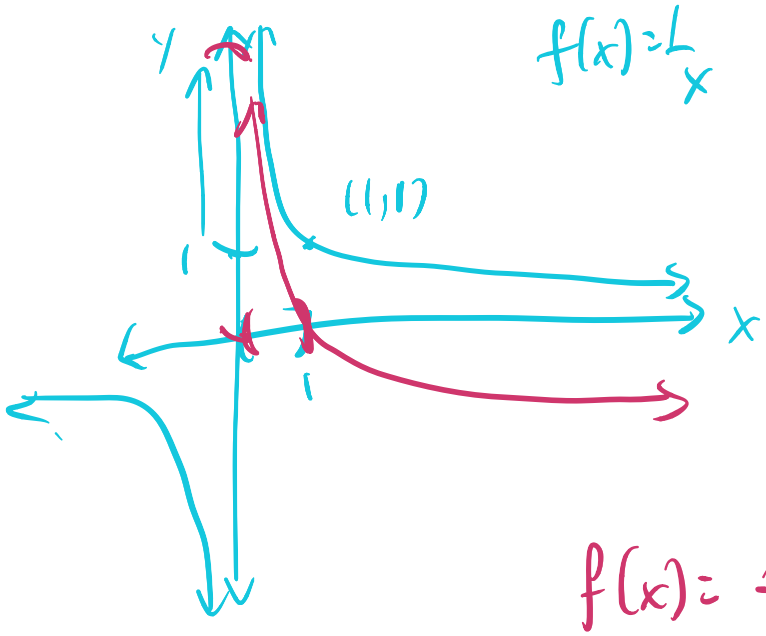
(a) Do $(0,1)$ and $\mathbb{R}_+ = (0,\infty)$ have the same cardinality? If so, either give an explicit bijection (and prove that it is a bijection) or provide an injection from $(0,1)$ to $(0,\infty)$ and an injection from $(0,\infty)$ to $(0,1)$ (so that by Cantor-Bernstein theorem the two sets will have the same cardinality). If not, then prove that they have different cardinalities.

(b) Is the set of strings over the English alphabet countable? (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.) If so, then provide a method for enumerating the strings. If not, then use a diagonalization argument to show that the set is uncountable.

(c) Consider the previous part, except now the strings are drawn from a countably infinite alphabet $\mathscr{A}$. Does your answer from before change? Make sure to justify your answer.

(a)





$f(x) = \frac{1}{x}$

$f(x) = 1 - \frac{1}{1-x}$

$1/x : (0,1) \to (1,\infty)$

$1/_{1-x} : (0,1) \to (1,\infty)$

$\frac{1}{x} - 1 : (0,1) \to (0,\infty)$

$$f(x) = \frac{1}{x} - 1$$

one-to-one: if $x \neq x'$, $f(x) \neq f(x')$

onto: for $y \in (0,\infty)$, find an $x \in (0,1)$ s.t. $f(x) = y$.

Yes, same cardinality.

(b). list length 1 strings, length 2 strings, ...

countable

infinite

(c). a
b
c
⋮
} infinite.

a
aa
aaa...

ℓ

k

1. list all strings of length at most 1 and using $\{a_1\}$.
2. list all strings of length at most 2 and using $\{a_1, a_2\}$
3. " " "3, and using $\{a_1, a_2, a_3\}$.

step $\leq \max(\ell, k)$. $\Rightarrow$ $\boxed{\text{countable}}$

$\underline{\mathbb{N} \times \mathbb{N}} \simeq \mathbb{N}$.

length 2 strings.

$\mathbb{N}(x)$ = all finite degree polynomials w/ coeff's in $\mathbb{N}$.

$\{ \mathbb{N} \setminus \{1\}, \mathbb{N} \setminus \{2\},$