Dis 06B - Counting II

- Hang out in my discord for a bit yesterday. Join today!

Reminder:

(factorial) $n! = n \times (n-1) \times \cdots \times 1$

(binom) $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

	sample w/ replacement	w/o
order matters	n^k	$\binom{n}{k} k!$
doesn't		$\binom{n}{k}$

?

Balls & Boxes: how many can I distribute k things & into n distinguishable boxes?

n balls $\quad \underbrace{o \; o \,|\, o \; | \cdots | \, o}_{k-1 \text{ dividers}} \quad \} k.$

$\sqcup_1 \quad \sqcup_2 \quad \sqcup_3 \quad \cdots \quad \sqcup_n$

$$\frac{(n+k-1)!}{n!\,(k-1)!} = \binom{n+k-1}{k-1}.$$

Combinatorial Proofs (later)

1 Count it

Let's get some practice with counting!

(a) How many sequences of 15 coin-flips are there that contain exactly 4 heads?

(b) An anagram of HALLOWEEN is any re-ordering of the letters of HALLOWEEN, i.e., any string made up of the letters H, A, L, L, O, W, E, E, N in any order. The anagram does not have to be an English word.
How many different anagrams of HALLOWEEN are there? (catherine)

(c) How many solutions does $y_0 + y_1 + \cdots + y_k = n$ have, if each y must be a non-negative integer?

(d) How many solutions does $y_0 + y_1 = n$ have, if each y must be a positive integer?

(e) How many solutions does $y_0 + y_1 + \cdots + y_k = n$ have, if each y must be a positive integer?

$-1 \quad -1 \cdots \quad -1 = (k-1)$

(a) $\underline{H}\ \underline{H}\ \underline{H}\ \underline{H}\ \underline{T} \ldots \underline{T}\ \underline{T}\ \underline{T}$ => 4 heads, 11 tails.

$\binom{15}{4}$ | all perms (anagrams) of 4 H's, 11 T's => $\frac{15!}{4!\,11!}$

(b) 1st treat all letters as unique => 9!

But L's and E's aren't, so divide by 2! for each.

total: $\boxed{\frac{9!}{2!\,2!}}$ ex: HAL_1L_2 => $\left.\begin{matrix} HAL_1L_2 \\ HAL_2L_1 \end{matrix}\right\} 2!$

(c) Balls and Boxes.

$$\underbrace{\underbrace{o\ o\ o}_{y_0} | \cdots | \underbrace{o}_{y_{k-1}} | \underbrace{o}_{y_k}}_{n \text{ balls}} = \binom{n+k}{k}$$

(d)

y_0	y_1
1	n-1
2	n-2
⋮	⋮
n-2	2
n-1	1

} n-1 sols.

(e) $y_0 \geq 1 \Rightarrow y_0' \geq 0$

$y_0' = y_0 - 1$

$y_0' + \cdots + y_k' = n-k-1$
(nonnegative)

$\binom{n-k-1+k}{k} = \binom{n-1}{k}$

(e). $y_0 + \cdots + y_k = n$ (positive, int).

$n = 6$

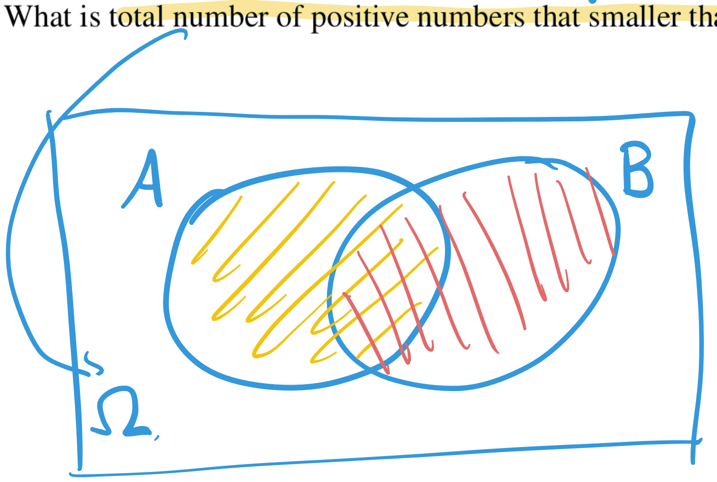
$$\underbrace{o \,\_\, o \,\_\, \mid o \,\_\, o \,\_\, o \,\_\, o \,\_\, \mid \,\_\, o}_{n \text{ balls}}$$

$$\binom{n-1}{k}.$$

2 Inclusion and exclusion

"universe"

What is total number of positive numbers that smaller than 100 and coprime to 100?



$|A \cup B| = |A| + |B| - |A \cap B|$.

1) when is a # coprime to 100?
2) what would the sets A & B be in this case?

1) => not divisible by 2 or 5.

2) complement => #s aren't coprime to 100.
-> mult. of either 2 or 5.

A = multiples of 2 = $\{2, 4, 6, \ldots, 98\}$.
$|A| = 49$

B = multiples of 5 = $\{5, 10, \ldots, 95\}$
$|B| = 19$.

$A \cap B$ = mult. of 10 = $\{10, 20, \ldots, 90\}$
$|A \cap B| = 9$.

$|A \cup B| = |A| + |B| - |A \cap B| = \underline{59}$.

So # that <u>are</u> coprime is $99 - 59 = \boxed{40}$