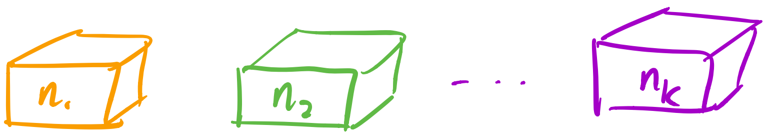
Dis 06A: Counting I

- Midterm scope, logistics out on Piazza
- HW 6 length has been reduced
- Staff Review Session this weekend (Sun 2-4pm), CSM + HKN to be announced.
- Topic specific review?
- Questions for today are slightly different: see tylerzhu.com/teaching/fa20
- look at my last semester's recaps.

Review

1^{st} Rule of Counting:



\# of ways to pick one item from each box $= n_1 \times n_2 \times \cdots \times n_k$.

Question: How many ways to rearrange n distinct books on a bookshelf?

Answer: $\underbrace{\underline{n} \times \underline{(n-1)} \times \underline{\cdots} \times \underline{1}}_{\text{factorial}} = \underline{n!}$

Question: How many ways can I choose k items from n w/o regards to order?

Answer:

$$\frac{n \times (n-1) \times \cdots \times (n-k+1)}{k!}$$

← # of ways to choose

← # of ways to rearrange.

$$= \frac{n!}{k!\,(n-k)!} = (n \text{ choose } k) = \binom{n}{k} \quad {}_nC_k$$

	sampling w/ replacement	w/o replacement	
order matters	n^k	$\frac{n!}{(n-k)!}$	(permutation)
doesn't matter	$\binom{n+k-1}{k-1}$	$\binom{n}{k}$	(combination)

picking k items from n

where . . .

1 Zerg Player

A Zerg player wants to produce an army to fight against Protoss in early game, and he wants to have a small army which consumes exactly 10 supply. And he has the following choices:

- Zerglings: consumes 1 supply $\# = x$
- Hydralisk: consumes 2 supply $\# = y$
- Roach: consumes 2 supply $\# = z$

How many different compositions can the player's army have? Note that Zerglings are indistinguishable, as are Hydralisks and Roachs.

$$x + 2y + 2z = 10$$

$$x + 2 \cdot y' = 10 \qquad (y' = y+z)$$

$$x = 2x'$$

$$\boxed{x' + y + z = 5} \rightarrow \binom{5+3-1}{3-1}$$

$$6 + 5 + 4 \cdots + 1 = \frac{6 \cdot 7}{2} = \boxed{21}$$

\# of Zerglings must be even.

x	y'	y, z	
0	5	5,0 4,1 ⋮ 0,5	} 6
2	4	4,0 ⋮ 0,4	} 5
4	3	3,0 ⋮ 0,3	} 4
6	2	⋮	
8	1		
10	0	0 0	21

$$\frac{20!}{10! \cdot 2^{10}} = \frac{\cancel{20} \cdot 19 \cdot \cancel{18} \cdot 17 \cdot \cancel{16} \cdot 15 \cdot \cancel{14} \cdot 13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10} \cdots}{\cancel{2 \cdot 10} \cdot \cancel{2 \cdot 9} \cdot \cancel{2 \cdot 8} \cdot \cancel{2 \cdot 7} \cdot \cancel{2 \cdot 6} \cdot \cancel{2 \cdot 5} \cdots}$$

$$= 19 \cdot 17 \cdot 15 \cdot 13 \cdot \cdots \cdot 3 \cdot 1$$

$$(S_3\ S_5)(S_7\ S_2)(S_6\ S_8)(S_4\ S_1)$$

$$\rightarrow (S_3\ S_5)(S_2\ S_7)(S_6\ S_8)(S_1\ S_4)$$

$$\rightarrow (S_1\ S_4)(S_2\ S_7)(S_3\ S_5)(S_6\ S_8)$$

2 Counting Practice

(a) If you shuffle two (identical) decks of cards together, you get a stack of 104 cards, where each different card type is included twice. How many different ways are there to order this stack of cards?

104 unique: $104!$

not unique though, 52 same pairs.

For 1 pair can have $A_1 \dots A_2$ or $A_2 \dots A_1$ (same!), divide by 2.

52 pairs, divide by 2^{52} -> $\boxed{\frac{104!}{2^{52}}}$

(b) How many different anagrams of GHOST are there if: (1) H is the right neighbor of G; (2) G is on the left of H (and not necessarily H's neighbor)?

(1) combine (GH), so 4 choices total -> $4!$

(2) G _ _ H _

$\binom{5}{2}$ - for G, H (G has to be b4 H)

$3!$ about the rest.

$\frac{5 \cdot 4}{2} \cdot 3! = \frac{5!}{2} = \underline{60}$

GOSHT

HOSGT.

-> $\frac{1}{2} \cdot 5!$

(c) There are 20 socks in a drawer, none of which match. How many different ways are there to pair up these socks? (Assume that any sock can be paired with any other sock.)

$20!$ ways to order socks

-> $10!$ ways to order pairs

$\boxed{\frac{20!}{10!} \cdot \frac{1}{2^{10}}}$

ordering of all pairs

order w/in pairs.

$(S_3\ S_2)$

$\underbrace{(S_2\ S_3)(S_5\ S_6)(S_{20}\ S_1)\dots}_{10!}$

$(S_5 S_6)\ (S_2\ S_3)$

$(S_1\ S_{20})(S_2\ S_3)$

$(S_4\ S_{..})(S_5\ S_6)$

Sol 2: $\binom{20}{2}\binom{18}{2}\binom{16}{2}\binom{14}{2}\cdots\binom{2}{2}/10!$

Sol 3: $\bigcirc\ \underline{19}\ |\ \bigcirc\ \underline{17}\ |\ \bigcirc\ \underline{15}\ |\ \bigcirc \cdots |\ \bigcirc\ \underline{1}.$

$= \underline{19 \times 17 \times 15 \cdots \times 3 \times 1.}$

2 Counting Practice (solution writeup: from sp20)

(a) If you shuffle two (identical) decks of cards together, you get a stack of 104 cards, where each different card type is included twice. How many different ways are there to order this stack of cards?

Essentially looking for # of permutations of $\underbrace{C_1 C_1 C_2 C_2 \ldots C_{52} C_{52}}_{\text{104 cards in total.}}$

104! ways to permute w/o regards to the indistinguishable duplicates,

For each pair, 2 equivalent ways they can be ordered $\Rightarrow$ $\boxed{\frac{104!}{2^{52}}}$

(b) How many different anagrams of GHOST are there if: (1) H is the right neighbor of G; (2) G is on the left of H (and not necessarily H's neighbor)?

(1) Glue G & H together. This is the same as finding anagrams of (GH), O, S, T. $\Rightarrow$ 4! ways.

(2) Anagrams where G is on the left of H are in 1-1 correspondence w/ those where G is on the right of H, (just flip G & H), so just $\frac{1}{2}$ · total $= \boxed{\frac{5!}{2}}$

- OR. *****. Choose 2 spots for G, H and put G in the first, H in the second. 3! ways to distribute the rest $\Rightarrow \binom{5}{2} \cdot 3! = \boxed{5!/2}$

(c) There are 20 socks in a drawer, none of which match. How many different ways are there to pair up these socks? (Assume that any sock can be paired with any other sock.)

$(S_4\ S_2)(S_6 S_7)(S_5 S_9)(S_3\ S_1)(S_{10} S_8)\ldots$

Sol 1. Given some permutation of the 20 socks, we can just pair up the 1st & 2nd, 3rd & 4th, ... But we need to disregard order (since we don't care how we created them), so divide by 10! for order b/w the pairs, and by 2 for the order w/in each pair

$\Rightarrow \boxed{\frac{20!}{10! \cdot 2^{10}}}$

Sol 2. I can also just pick socks two at a time w/o order, and then divide to account for order b/w pairs $\Rightarrow \boxed{\binom{20}{2}\binom{18}{2} \cdots \binom{2}{2} / 10!}$

Sol 3.

- Find a way to fix orderings when counting. For example, if we're trying to find the number of ways to pair up 20 different socks, create pairs by fixing the smallest numbered sock not yet chosen as the first sock. There is an unambiguous ordering both within each pairs and between all of the pairs; the smaller number in each pair comes first, and all of the pairs are ordered by increasing smallest sock. This let's us compute the result as $19 \times 17 \times \cdots \times 1$. The same idea applies for finding the number of permutations of GHOST where G comes before H.