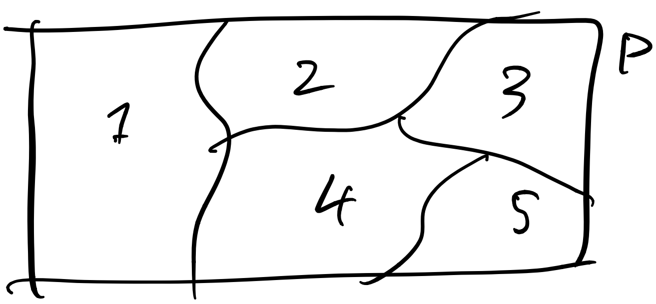
Dis 01A: Proofs

- Fill out bit.ly/tyler-dis1a
- tylerzhu.com/teaching/fa20
- Vit 1, HW due Friday
- Tmr's Lecture on Induction, Note 3
- AI's: Sathvik, Catherine Gai
- Demo oh.eecs70.org

Review

- Direct Proof: show $P \implies Q$ where P is a truth and Q is our claim.
- Contrapositive: for a statement $P \implies Q$, prove $\neg Q \implies \neg P$.
- Contradiction: to prove a claim P, assume for the sake of contradiction that $\neg P$ is true. Show this implies $R \wedge \neg R$ (for some R), contradiction. Hence P is true.
- By cases: To show P is true in general, we prove P in separate cases, which in combination cover all possible cases (i.e. cases are a partition).



1 Proof Practice

(a) Prove that $\forall n \in \mathbb{N}$, if n is odd, then n^2+1 is even. (Recall that n is odd if $n = 2k+1$ for some natural number k.)

n is odd $\Rightarrow$ $n = 2k+1$ for $k \in \mathbb{N}$

direct.

$$n^2+1 = (2k+1)^2+1 = (4k^2+4k+1)+1$$
$$= 4k^2+4k+2$$
$$= \underline{2(2k^2+2k+1)}$$
, must be even.

(b) Prove that $\forall x, y \in \mathbb{R}$, $\min(x,y) = (x+y-|x-y|)/2$. (Recall, that the definition of absolute value for a real number z, is

$$|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$$

Case 1: $x > y$: $\min(x,y) = \dfrac{x+y-|x-y|}{2} = \dfrac{x+y-(x-y)}{2}$

$\Rightarrow \min(x,y) = y$ $\quad = y$ ✓

and $|x-y| = x-y$, so

Case 2: $x \leq y$: $\min(x,y) = \dfrac{x+y-|x-y|}{2} = \dfrac{x+y-(y-x)}{2}$

$= x$ ✓.

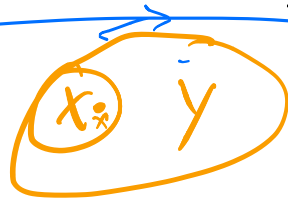
(c) Suppose $A \subseteq B$. Prove $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. (Recall that $A' \in \mathscr{P}(A)$ if and only if $A' \subseteq A$.)

$A = \{1,2\}$

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

= set of all subsets

$\boxed{X \subseteq Y \text{ iff } \forall x \in X, x \in Y}$.



$A' \in \mathcal{P}(A) \Rightarrow A' \subseteq A$.

But $A \subseteq B$, so $A' \subseteq B$, and hence $\underline{A' \in \mathcal{P}(B)}$.

2 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

$\Rightarrow$: $x \in f^{-1}(A \cup B)$ $\Rightarrow$ $x \in f^{-1}(A) \vee f^{-1}(B)$

$\Rightarrow$ $f(x) \in A \cup B$ — by def. of pre-image

$\Rightarrow$ $f(x) \in A \vee f(x) \in B$ — def of union

$\Rightarrow$ $x \in f^{-1}(A) \vee x \in f^{-1}(B)$ — by def of pre-image.

$\Rightarrow$ $x \in f^{-1}(A) \vee f^{-1}(B)$ — def of union. ∎

(a) $f^{-1}(A \cup B) \overset{?}{=} f^{-1}(A) \cup f^{-1}(B)$.

$f: \mathbb{Z} \to \mathbb{Z}$

$f(x) = 2x$.

$f(\{1,2\}) = \{2,4\}$

$f^{-1}(\{4,5,6\}) = \{2,3\} \subseteq A$

$f(x) = 3$?

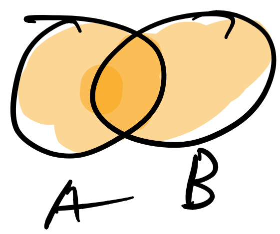
$f: A \longrightarrow B$

$Y \subseteq B$

$\Rightarrow$: $x \in f^{-1}(A \cup B)$ $\Rightarrow$ $x \in f^{-1}(A) \cup f^{-1}(B)$

def: $f^{-1}(A \cup B) = \{x \mid f(x) \in A \cup B\}$

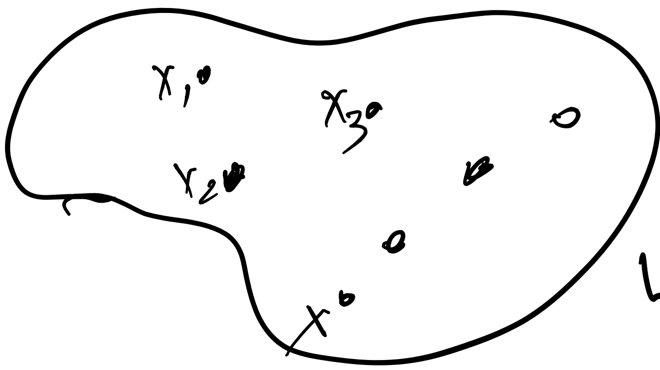
$f(x) \in A \cup B$ $\Rightarrow$ $f(x) \in A$ or $f(x) \in B$



$\Rightarrow$ $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$ $\Rightarrow$ $x \in f^{-1}(A) \cup f^{-1}(B)$

$f^{-1}(Y) = \{x \mid f(x) \in Y\}$.

$f^{-1}(A) = \{x \mid f(x) \in A\}$.



$f^{-1}(A)$

$f(x_1) \in A$

$f(x_2) \in A$

$= 8 \; x.$